Analytical Modeling of Dynamic Behavior of Composite Materials

* Types of dynamic loading
  - Impulsive (transient)
  - Oscillatory (steady state)

* Dynamic response characteristics
  - Wave propagation
  - Vibrations (standing waves)

* Use of effective modulus theory (EMT)
  - Criteria for use of EMT are met by most practical vibration problems since wavelength $\gg$ scale of inhomogeneity, but this may not be true for high frequency wave propagation
  - Wave propagation problems may also involve complex reflection/refraction effects due to multiple interfaces, and wave speed depends on direction, unlike isotropic materials

Concept of an Effective Modulus of an Equivalent Homogeneous Material.

Heterogeneous composite under varying stresses and strains

Equivalent homogeneous material under average stresses and strains
Two Criteria for use of Effective Modulus Theory in Dynamic Loading of Composites

\[ d = \text{scale of inhomogeneity in composite} \]

\[ L \gg d \quad \text{Averaging dimension} \]

\[ \lambda \gg d \quad \text{Wavelength of dynamic stress distribution} \]

Important modes of vibration of structural and machine elements

- Flexural
- Torsional
- Longitudinal
Natural frequencies of flexural and torsional modes of vibration are usually in the range of typical excitation frequencies, but longitudinal modes are usually well above this range.

**Review of continuous system models**

**Examples of one-dimensional wave equations**

**Longitudinal waves and vibrations in a linear elastic homogeneous isotropic bar**

\[
\frac{\partial^2 u}{\partial t^2} + \frac{P}{\rho A} = 0
\]

Equation of motion: \( \sum F_x = (P + \frac{\partial P}{\partial x}) - \mathcal{F} = \rho A \frac{\partial^2 u}{\partial t^2} \)

where \( u \) = displacement, \( a \) = area
\( \rho \) = mass density

\( P = \sigma = AE \varepsilon \)

Thus, \( \frac{\partial^2 u}{\partial x^2} + \frac{P}{\rho A} \frac{\partial^2 u}{\partial t^2} = 0 \)

For linear elastic isotropic bar, \( \frac{\partial}{\partial t} AE = 0 \), then:

\( \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{2E} \frac{\partial^2 u}{\partial t^2} \)

or \( c = \sqrt{\frac{E}{\rho}} = \text{wave speed} \)
Note: If the criteria for the use of a non-linear medium have been met, and the bar is a specially orthotropic composite, these equations would also be valid for the case of longitudinal (E=Eₓ) or transverse (E=Eᵧ) wave propagation, and \( \rho \) is the composite density.

Other examples of 1-D Wave Equation:
- Torsional oscillations of a string
  \[ c^2 \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial t^2} \]
  \( c = \sqrt{\frac{T}{\rho}} \)
  \( \psi \): displacement
- Torsional oscillations of circular bar
  \[ c^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t^2} \]
  \( c = \sqrt{\frac{G\theta}{\rho}} \)
  \( G \): shear modulus
  \( \theta \): angle of twist
- Acoustic plane waves
  \[ c = \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \]
  \( c = \sqrt{\frac{\rho}{\mu}} \)
  \( \phi \): pressure

Solutions of 1-D Wave Equation

1. d'Alembert solution: \( u(x,t) = p(x+ct) + p(x-ct) \)

   where \( p(x+ct) \) represents a wave travelling to the left with velocity \( c \) (i.e., point \( x = x+ct = \text{constant} \)),
   and \( p(x-ct) \) represents a wave travelling to the right with velocity \( c \).

   Similarly, \( p(x+ct) \) represents a wave travelling to the right with velocity \( c \).

   For a sine wave,
   \[ u(x,t) = A \sin \left( \frac{\pi x}{\lambda} (x+ct) \right) + A \sin \left( \frac{\pi x}{\lambda} (x-ct) \right) \]
   where \( \lambda \): wavelength (Note: \( \lambda \approx a \) for EMT)

   Alternatively, \( u(x,t) = A \sin \left( \frac{2\pi x}{\lambda} ct \right) + A \sin \left( \frac{2\pi x}{\lambda} ct \right) \)
   where \( k = \frac{1}{\lambda} \): wave number = no. of waves/unit distance
   \( \omega = c(\frac{2\pi}{\lambda}) \): frequency of wave
Using trigonometric identities,
\[ u(x,t) = 2A \sin \omega \pi x \cos \omega t \]
which represents a standing wave of profile \( 2A \sin \omega \pi x \) which oscillates with frequency \( \omega \).

Generally, the combined wave motion in opposite directions is caused by reflections from boundaries. Thus, wave propagation without reflection will not lead to a standing wave, or vibration.

2. A similar solution is found by using separation of variables
\[ u(x,t) = U_0(x)f(t) \]
substituting this solution in the 4-D wave equation.

\[ \frac{d^2 U}{d x^2} + \frac{1}{c^2} \frac{d^2 U}{d t^2} = \frac{1}{F} \frac{d^2 F}{d x^2} = - \omega^2 \]
Thus, the two resulting equations are
\[ \frac{d^2 F}{d t^2} + \omega^2 F = 0 \]
and
\[ \frac{d^2 U}{d x^2} + \left( \frac{\omega}{c} \right)^2 U = 0 \]
and the solutions are of the form
\[ F = A \sin \omega t + B \cos \omega t \]
\[ U = A_1 \sin \frac{\omega}{c} xt + B_2 \cos \frac{\omega}{c} xt \]
where \( A_1 \) and \( B_1 \) depend on the initial conditions and \( A_2 \) and \( B_2 \) depend on the boundary conditions.
For a bar fixed on both ends:

\[ u(0,t) = u(L,t) = 0 \]

Substitution of the boundary conditions in the solution leads to the ordinary equation

\[ \sin \frac{nk}{L} = 0 \]

which has an infinite number of solutions \( \omega_n \) such that

\[ \frac{nk}{L} = n\pi \quad n = 1, 2, 3, \ldots, \infty \]

where \( \omega_n \) = natural frequency (mass) = \( 2\pi f_0 \)

\( f_0 \) = natural frequency (frequency)

Thus,

\[ f_n = \frac{nk}{2L} = \frac{n\pi}{2L} \sqrt{E/\rho} \]

For the \( n^\text{th} \) mode of vibration,

\[ u_n(x,t) = A_n \sin \omega_n t \cos \frac{nk}{L} + B_n \cos \omega_n t \sin \frac{nk}{L} \]

where \( A_n = A_n A_1 \), \( B_n = B_n B_1 \)

The mode shape for the \( n^\text{th} \) mode is given by the eigenfunction

\[ \psi_n(x) = \sin \frac{nk}{L} \]

The general solution is the superposition of all modes

\[ u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t \cos \frac{nk}{L} + B_n \cos \omega_n t \sin \frac{nk}{L}) \]

Examples of first three modes:

\[ n=1 \]

1. Mode shape:
2. \( f_1 = \frac{1}{2L} \sqrt{E/\rho} \)

\[ n=2 \]

1. Mode shape:
2. \( f_2 = \frac{1}{L} \sqrt{E/\rho} \)

\[ n=3 \]

1. Mode shape:
2. \( f_3 = \frac{3}{2L} \sqrt{E/\rho} \)
Mode shapes and frequencies for longitudinal vibration of a bar with both ends fixed

\[ u(x) \]
\[ n = 1 \]
\[ f_n = \frac{\pi^2}{4L^2} \cdot \lambda_n = 2L \]
\[ u(x) \]
\[ n = 2 \]
\[ f_n = \frac{\pi^2}{2L^2} \cdot \lambda_n = L \]
\[ u(x) \]
\[ n = 3 \]
\[ f_n = \frac{\pi^2}{3L^2} \cdot \lambda_n = \frac{2L}{3} \]

Examples from *Formulas for Natural Frequency and Mode Shape*, by R. D. Blevins, 1979
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5. Fixed-Fixed

\[
\omega_n = \sqrt{\frac{k}{m}} \quad \text{with} \quad m = \frac{1}{2} L^2 \quad (n = 1, 2, 3, \ldots)
\]

See Fig. 3-21 for \( \omega_n \).

6. Fixed-Free

\[
\omega_n = \sqrt{\frac{k}{m}} \quad \text{with} \quad m = \frac{1}{4} L^2 \quad (n = 1, 2, 3, \ldots)
\]

See Fig. 3-22 for \( \omega_n \).

7. Free-Free

\[
\omega_n = \sqrt{\frac{k}{m}} \quad \text{with} \quad m = \frac{1}{3} L^2 \quad (n = 1, 2, 3, \ldots)
\]

See Fig. 3-23 for \( \omega_n \).

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This is a review article which summarizes a number of papers, one of which deals with measurement and analysis of wave velocities in unidirectional composites.

**Experimental Program:**

Tested bidirectional stainless steel fibers in an epoxy matrix. Long bar specimens with square fiber arrays were fabricated as shown below:

\[
\frac{d}{L} \geq 250
\]

**Observations:**

\[
V_f = 0.13 \times \sqrt{e} \quad (e < 0.4)
\]
Experimental Apparatus:

Stress waves introduced into long bar specimens by using a gas gun to propel short epoxy striker voa against the end of the specimen. The resulting stress wave was much greater in wave length than the diameter of the bar, \( \lambda \gg d \), thus effective modulus theory is valid.

Wave velocities by measuring time required for a given point on the wave to travel from one strike zone to the other:

\[
C = \frac{dL}{\Delta t}
\]

where \( dL \) = distance between stress gauges
\( \Delta t \) = elapsed time

\[
\text{From the solution to the one-dimensional wave equation, the wave speed is:}
\]

\[
C = \sqrt{\frac{E}{\rho}}
\]

For longitudinal specimens, the modulus of elasticity is \( E = E_1 \), where

\[
E_c = E_f v_f + E_m v_m
\]

and the density is

\[
\rho = \rho_f v_f + \rho_m v_m
\]

so the wave speed is

\[
C_c = \sqrt{\frac{E_c}{\rho}} = \sqrt{\frac{E_f v_f + E_m v_m}{\rho_f v_f + \rho_m v_m}}
\]

For transverse specimens, the modulus of elasticity is \( E = E_2 \), where \( E_c \) is given by H Albuquerque equations:

\[
E_c = E_m \left[ \frac{1 + \eta v_f}{1 - \eta v_f} \right]
\]

where \( \eta = \frac{(E_f/E_m) - 1}{(E_f/E_m) + 2} \)

\[
\text{and } \beta = 2 \text{ for square array of circular fibers.}
\]
Thus the transverse wave speed is

\[ c_t = \sqrt{\frac{E_t}{\rho}} = \sqrt{\frac{E_t}{(1 + 3\mu)\nu_t}} \left( \frac{1 - 2\nu_t}{(1 - \nu_t)} \right) \]

The measured and predicted wave speeds are shown below.

Note that, for \( \nu_t = 0 \), \( c = c_t \) and for \( \nu_t = 100 \), \( c = c_p \). For low \( \nu_t \), \( c_t < c_p \).


Tested the limits of effective modulus theory by measuring wave speeds of ultrasonic stress waves propagating through particle reinforced composites.

\[ a = \text{particle diameter} \]

Four types of composites tested:
1. Lead particles in epoxy matrix
2. Glass particles in epoxy matrix
3. Steel particles in epoxy matrix
4. Glass microballoons in plexiglas matrix
Conclusion: Effective modulus theory can be used to predict the wave speed of stress waves in particle reinforced composites long as the wavelength, $\lambda$, is much greater than the particle size, $a$. 

$\lambda \gg a$