Deflections of Beams

Recall moment – curvature equation

\[ \frac{1}{\rho} = \frac{M}{EI} \]

From calculus, recall that the radius of curvature of a function \( y(x) \) is

\[ \rho = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{-3/2} \]

\[ \rho \approx \frac{1}{\frac{d^2y}{dx^2}} \]

Where \( y(x) \) is a function of \( x \) which in this case is a curve describing the neutral surface of the beam and \( x \) is the length direction of beam.

for small slopes, \( \frac{dy}{dx}, \left( \frac{dy}{dx} \right)^2 \ll 1 \) and

\[ \rho \approx \frac{1}{\frac{d^2 y}{dx^2}} \]
\[ \therefore \frac{d^2 y}{dx^2} = \frac{M}{EI} \]

which is the differential equation of the elastic curve of the beam.

What is thickest belt that can be used without exceeding a stress of 280MPa?
Recall bending strain

\[ \varepsilon_x = -\frac{y}{\rho} \]

\[ \varepsilon_{x_{\text{max}}} = -\frac{c}{\rho} = -\frac{t}{2\rho} \left( c = y_{\text{max}} = \pm \frac{t}{2} \right) \]

Stress \( \sigma_x = E\varepsilon_x = \frac{Ey}{\rho} \)

\[ \sigma_{x_{\text{max}}} = \pm \frac{Et}{2\rho} \]

\[ . \quad t = \frac{2\rho\sigma_{x_{\text{max}}}}{E} = \frac{2(150)(10^3)(280)\times10^6}{200\times10^6} \]

\[ t = 0.00042 \quad m = 0.42 \text{ mm} \]

In this case, the radius of curvature , \( \rho \), is known, but generally the deflection \( y(x) \) is more useful than the curvature, so we now focus on the deflection \( y(x) \).

to find deflection, \( y(x) \),

1) Find expression for \( M(x) \)

2) Integrate \( \frac{d^2y}{dx^2} = \frac{M}{EI} \) twice to get \( y(x) \)

3) Substitute in B.C’s to find integration constants
Example:

\[ \sum M_{\text{cut}} = M - Px + PL = 0 \]

\[ \therefore M = P(x - L) \]

\[ \therefore EI \frac{d^2 y}{dx^2} = P(x - L) \]

\[ EI \frac{dy}{dx} = P \left( \frac{x^2}{2} - Lx \right) + C_1 \]

\[ EI \cdot y = P \left( \frac{x^3}{6} - \frac{Lx^2}{2} \right) + C_1 x + C_2 \]

B.C's:

\[ y(0) = 0 \quad \therefore C_2 = 0 \]

\[ \frac{dy}{dx}(0) = 0 \quad \therefore C_1 = 0 \]

\[ \therefore y = \frac{P}{EI} \left( \frac{x^3}{6} - \frac{Lx^2}{2} \right) \]

\[ y_{\text{max}} = y(L) = \frac{P}{EI} \left( \frac{L^3}{6} - \frac{L^3}{2} \right) = -\frac{PL^3}{3EI} \]
Example: Multiple sections

From statics,

1. in \( 0 \leq x \leq a \), \( M = P(x-a) \)
2. in \( a \leq x \leq L \), \( M = 0 \)

Thus for section 1:

\[
\begin{align*}
\frac{d^2 y_1}{dx^2} &= \frac{P}{EI}(x-a) \\
\frac{dy_1}{dx} &= \frac{P}{EI} \left( \frac{x^2}{2} - ax \right) + C_1 \\
y_1 &= \frac{P}{EI} \left( \frac{x^3}{6} - \frac{a x^2}{2} \right) + C_1 x + C_2
\end{align*}
\]

and for section 2:

\[
\begin{align*}
\frac{d^2 y_2}{dx^2} &= 0 \\
\frac{dy_2}{dx} &= C_3 \\
y_2 &= C_3 x + C_4
\end{align*}
\]
B.C’s:
\[ y_2(a) = C_3(a) + C_4 = y_1(a) = \frac{P}{EI} \left( \frac{a^3}{6} - \frac{a^3}{2} \right) \]
\[ \frac{dy_2}{dx}(a) = C_3 = \frac{dy_1}{dx} = \frac{P}{EI} \left( \frac{a^2}{2} - a^2 \right) \]
\[ \therefore C_3 = -\frac{Pa^2}{2EI}, \quad C_4 = \frac{Pa^2}{6EI} \]

Thus,
\[ y(x) = -\frac{Pa^2x}{2EI} + \frac{Pa^3}{6EI} \quad \text{and} \quad y_{\text{max}} = y(L) = -\frac{Pa^2L}{2EI} + \frac{Pa^3}{6EI} \]

Singularity Function Method for Beam Deflections

When multiple loading is present, the integration method becomes very cumbersome because multiple sets of equations must be integrated and the corresponding integration constants must be found from B.C’s

Example:
for $0 < x < x_1$:

\[ + \sum M_{\text{cut}} = M - R L x = 0 \]
\[ \therefore M = R L x \]

for $x_1 < x < x_2$:

\[ + \sum M_{\text{cut}} = M + P (x - x_2) - R L x = 0 \]
\[ \therefore M = R L x - P (x - x_2) \]

Similarly, for the remaining sections of the beam,

for $x_2 < x < x_3$:

\[ M = R L x - P (x - x_2) + M_0 \]

for $x_3 < x < L$:

\[ M = R L x - P (x - x_2) + M_0 - \frac{w}{2} (x - x_3)^2 \]

for each section

\[
\frac{d^2 y}{dx^2} = \frac{M}{E I}
\]

We have to integrate each of these four equations twice and so 8 B.C’s are required to find $y(x)$.

However, with singularity functions, we only need to write one equation and integrate twice for the entire beam.

For above beam the equation below applies for all $x$:

\[ M = R L (x - 0)^1 - P (x - x_1)^1 + M_0 (x - x_2)^0 - \frac{w}{2} (x - x_3)^2 \]
where the pointed brackets have the following meanings

\[
\langle x-a \rangle^n = \begin{cases} 
(x-a)^n & \text{if } x > a \\
0 & \text{if } x < a 
\end{cases}
\]

where \( n \geq 0 \)

and \( \langle x-a \rangle^0 = \begin{cases} 
1 & \text{if } x > a \\
0 & \text{if } x < a 
\end{cases} \)

Procedure for using singularity functions to find deflection \( y(x) \),

1. Make imaginary cut in last distinct region of beam

2. Find \( M \) from statics.

3. Modify equation for \( M \) using singularity functions so that it applies for entire beam.

4. Integrate twice to get \( y(x) \). Singularity function integrates just like algebraic expression with parentheses. That is,

\[
\int \langle x-a \rangle^n \, dx = \frac{\langle x-a \rangle^{n+1}}{n+1} + C \quad \text{for } n \geq 0
\]
So, for this example,

\[
EI \frac{d^2y}{dx^2} = R_i \langle x-0 \rangle^1 - P \langle x-x_i \rangle^1 + M_o \langle x-x_i \rangle^0 - \frac{w}{2} \langle x-x_i \rangle^2
\]

\[
EI \frac{dy}{dx} = R_i \frac{\langle x \rangle^2}{2} - P \frac{\langle x-x_i \rangle^2}{2} + M_o \langle x-x_i \rangle^1 - \frac{w}{6} \langle x-x_i \rangle^2 + C_i
\]

\[
EI \quad y = R_i \frac{\langle x \rangle^3}{6} - \frac{P}{6} \langle x-x_i \rangle^3 + \frac{M_o \langle x-x_i \rangle^2}{2} - \frac{w}{24} \langle x-x_i \rangle^4 + C_i x + C_2
\]

B.C’s (do not forget the definition of pointed bracket here)

\[y(0) = 0 = C_2\]

\[y(L) = 0 = \frac{RL^3}{6} - \frac{P}{6} (L-x_i)^3 + \frac{M_o}{2} (L-x_i)^2 - \frac{w}{24} (L-x_i)^4 + C_i L\]

\[\therefore C_i = - \frac{RL^2}{6} + \frac{P}{6L} (L-x_i)^3 - \frac{M_o}{2L} (L-x_i)^2 + \frac{w}{24L} (L-x_i)^4\]

---

Graphical representations of singularity function

1. \(\langle x-a \rangle^0\)
2. \(\langle x-a \rangle^1\)
3. \(\langle x-a \rangle^2\)
Find (a) deflection at load $P$
(b) maximum deflection

\[ \sum M_{cut} = M + P(x - L) - Px + \frac{PL}{4} = 0 \]

\[ M = -\frac{PL}{4} + Px - P(x - L) \]

in terms of singularity functions

\[ EI \frac{d^2y}{dx^2} = M = -\frac{PL}{4} (x - 0)^0 + P(x - 0)^1 - P(x - L) \]

\[ EI \frac{dy}{dx} = -\frac{PL}{4} (x - 0)^1 + \frac{P}{2} (x - 0)^2 - P(x - L)^2 + C_1 \]

\[ EI \ y = -\frac{PL}{8} (x - 0)^2 + \frac{P}{6} (x - 0)^3 - \frac{P}{3} (x - L)^3 + C_1 x + C_2 \]

\[ y(0) = 0 = C_2 \]

\[ \frac{dy}{dx}(0) = 0 = C_1 \]
(a) \[ Y(L) = \frac{1}{EI} \left( -\frac{PL^3}{8} + \frac{PL^3}{6} \right) = \frac{PL^3}{24EI} \]

(b) check for possible max., min., or inflection point by setting slope \( \frac{dy}{dx} = 0 \)

\[ \frac{dy}{dx} = \frac{1}{EI} \left( -\frac{PL}{4}x + \frac{Px^2}{2} - \frac{P}{2}(x - L)^2 \right) = 0 \]

check in \( 0 \leq x \leq L \):

\[ \frac{dy}{dx} = \frac{1}{EI} \left( -\frac{PL}{4}x + \frac{Px^2}{2} \right) = 0 \]

or \( x \left( \frac{x}{2} - \frac{L}{4} \right) = 0 \) \( \therefore x = 0, \frac{L}{2} \)

check

\[ y\left( \frac{L}{2} \right) = \frac{1}{EI} \left( -\frac{PL}{8} \left( \frac{L}{2} \right)^2 + \frac{P}{6} \left( \frac{L}{2} \right)^3 \right) = -\frac{PL^3}{96EI} \]

check

\[ y\left( \frac{3L}{2} \right) = \frac{1}{EI} \left[ -\frac{PL}{8} \left( \frac{3L}{2} \right)^2 + \frac{P}{6} \left( \frac{3L}{2} \right)^3 - \frac{P}{3} \left( \frac{3L}{2} - L \right)^3 \right] \]

\[ = \frac{25}{96} \frac{PL^3}{EI} \]

\( = y_{max} \)
Deflection curve:

Note that the condition \( \frac{dy}{dx} = 0 \) does not necessarily apply at points of maximum deflection. For example, at the tip of the beam where \( x = \frac{3L}{2}, \quad y = y_{\text{max}} = \frac{25}{96} \frac{PL^3}{EI} \) but \( \frac{dy}{dx} \neq 0 \) there. The condition \( \frac{dy}{dx} = 0 \) only defines a local maximum, minimum or inflection point, not necessarily a global maximum such as

\( y_{\text{max}} \quad \text{at} \quad x = \frac{3L}{2} \).
Sectionally continuous loads:

A distributed load cannot always be expressed by a single singularity function for all values of x. This is particularly true when the distributed load stops at some intermediate position other than the end of beam.

Example:

\[ w = \begin{cases} \frac{wL}{8} & \text{for } 0 < x < \frac{L}{2} \\ \frac{3wL}{8} & \end{cases} \]

\[ R_x = wx \]
\[ + \] \[ \sum M_{\text{cut}} = M + \frac{wx^2}{2} - \frac{3wL}{8} x = 0 \]

\[ \therefore M = -\frac{wx^2}{2} + \frac{3wL}{8} x = 0 \]

Singularity function: \[ M = -\frac{w}{2} (x - 0)^2 - \frac{3wL}{8} (x - 0) \]

But the load stops at \( x = \frac{L}{2} \), while the singularity function continues for all \( x \), thus, we need to cancel \( w \) beyond \( x = \frac{L}{2} \).

This can be done by putting in a fictitious upward load of \( w \) beyond \( x = \frac{L}{2} \) as shown below.
Now for $x > L/2$:

\[
\sum M_{ciw} = M + \frac{wx^2}{2} - \frac{w}{2} \left( x - \frac{L}{2} \right)^2 - \frac{3wL}{8} \quad x = 0
\]

Singularity function:

\[
M = -\frac{w}{2} \left( x - 0 \right)^2 + \frac{w}{2} \left( x - \frac{L}{2} \right)^2 + \frac{3wL}{8} x
\]

This cancels out the effect of $w$ beyond $x = L/2$
Using singularity functions, find the equation for the deflection at any distance \( x \) along the beam.

\[ \sum M_{cut} = M + \frac{wL^2}{2} + wL^2 - wLx + \frac{w}{2}(x-L)^2 - \frac{w}{2}(x-2L)^2 = 0 \]

\[ M = -wL^2 + wLx + \frac{wL^2}{2} - \frac{w}{2}(x-L)^2 + \frac{w}{2}(x-2L)^2 \]
In terms of singularity functions:

\[
EI\frac{d^2y}{dx^2} = M = -wL + wL(x-0) - \frac{wL^2}{2} - \frac{w(x-L)}{2} + \frac{w(x-2L)^2}{2}
\]

\[
EI\frac{dy}{dx} = -wE \frac{x}{x-0} \frac{wL^2}{2} - \frac{w(x-L)}{6} + \frac{w(x-2L)^3}{6} + C_1
\]

\[
EI \ y = \frac{wE x^2}{2} + \frac{wL}{6} \frac{(x-0)^3}{x-0} - \frac{wE}{4} \frac{(x-L)^2}{x-L} - \frac{w}{24} \frac{(x-L)^4}{x-L} + \frac{w}{24} \frac{(x-2L)^4}{x-2L} + C_2
\]

B.C’s

\[
y(0) = 0 \quad \therefore C_2 = 0
\]

\[
\frac{dy}{dx}(0) = 0 \quad \therefore C_1 = 0
\]

\[
\therefore y = \frac{1}{EI} \left[ -\frac{wE x^2}{2} + \frac{wL}{6} (x-0)^3 - \frac{wE}{4} (x-L)^2 - \frac{w}{24} (x-L)^4 + \frac{w}{24} (x-2L)^4 \right]
\]

Superposition Method for Beam Deflections

In a linear elastic structure under combined loading, the total deflection at any point is equal to the sum of the deflections due to each of the loads. The same is true for slopes. Thus, we can make use of tables such as in Appendix A, Table A-19, to find deflections and slopes in beams under multiple loadings.
## TABLE A.19 Beam Deflections and Slopes

<table>
<thead>
<tr>
<th>Case</th>
<th>Load and Support (Length L)</th>
<th>Slope at End (+ ( \theta ))</th>
<th>Maximum Deflection (+ upward)</th>
<th>Equation of Elastic Curve (+ upward)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( P ) at middle</td>
<td>( \theta = \frac{Pz^2}{EIT} ) at ( z = L )</td>
<td>( y_{max} = \frac{Pz^3}{3EI} ) at ( z = L )</td>
<td>( y = \frac{Pz^2}{6EI}(3L - z) )</td>
</tr>
<tr>
<td>2</td>
<td>( P ) at left end</td>
<td>( \theta = \frac{Pz^2}{EI} ) at ( z = L )</td>
<td>( y_{max} = \frac{Pz^3}{6EI} ) at ( z = L )</td>
<td>( y = \frac{Pz^2}{24EI}(3L^2 - 4Lz + L^2) )</td>
</tr>
<tr>
<td>3</td>
<td>( P ) at right end</td>
<td>( \theta = \frac{Pz^2}{2EI} ) at ( z = L )</td>
<td>( y_{max} = \frac{Pz^3}{6EI} ) at ( z = L )</td>
<td>( y = \frac{1}{12EI}(8L^3 - 10Lz^2 + 5L^2 - z^4) )</td>
</tr>
<tr>
<td>4</td>
<td>( M ) at middle</td>
<td>( \theta = \frac{Mz}{EI} ) at ( z = L )</td>
<td>( y_{max} = \frac{Mz^2}{2EI} ) at ( z = L )</td>
<td>( y = \frac{Mz^2}{2EI} )</td>
</tr>
<tr>
<td>5</td>
<td>( P ) at left end</td>
<td>( \theta_1 = \frac{Pz^3}{6EI} ) at ( z = 0 )</td>
<td>( y_{max} = \frac{Pz^3}{6EI} ) at ( z = \sqrt{L^2 - z^2}/2 )</td>
<td>( y = \frac{Pz^3}{6EI}(z^2 - \theta^2) ) at ( 0 \leq z &lt; a )</td>
</tr>
<tr>
<td>6</td>
<td>( P ) at right end</td>
<td>( \theta_1 = \frac{Pz^3}{6EI} ) at ( z = 0 )</td>
<td>( y_{max} = \frac{Pz^3}{6EI} ) at ( z = 0 )</td>
<td>( y = \frac{Pz^3}{4EI}(z^2 - 4z^2) ) at ( 0 \leq z \leq L/2 )</td>
</tr>
</tbody>
</table>
Example:

Find displacement at point B

\[(y_B)_{total} = (y_B)_{w0} + (y_B)_P + (y_B)_M\]
Case 2

\[(y_B)_{w_0} = -\frac{w_0 L^4}{8EI}\]

Case 1

\[(y_B)_P = -\frac{PL^3}{3EI}\]

Case 4

\[(y_B)_{M_0} = \frac{M_0 L^2}{2EI}\]

\[(y_B)_{total} = \frac{w_0 L^4}{8EI} - \frac{PL^3}{3EI} + \frac{M_0 L^2}{2EI}\]

Superposition method in analysis of overhung shaft

Rigid bearings

Shaft, EI

\[P = \text{load from gear on shaft}\]

Model:

\[y_c\]
Superposition:

\[ y_C = y_{c1} + y_{c2} \]

Where \( y_{c1} = \) deflection at C due to deflection of BC with B fixed (cantilever)

\[ y_{c1} = -\frac{PL^3}{3EI} \]

Case 1: in Appendix A

Where \( y_{c2} = \) deflection at C due to rotation at B without deflection in BC

\[ y_{c2} = \theta_b L \]

Case 8:

\[ \therefore y_C = y_{c1} + y_{c2} = \frac{PL^3}{3EI} - \frac{2PL^3}{3EI} = -\frac{PL^3}{EI} \]
Determine the deflection at right end (c)

\[ M_0 = \frac{wL^2}{2} \]

Superposition:

\[ y_C = y_1 + y_2 + y_3 + y_4 + y_5 \]

where

\[ y_1 = -\frac{w(2L)^4}{8EI} \quad \text{(Case 2)} \]

\[ y_2 + y_3 = \frac{wL^4}{8EI} + \theta_x L = \frac{wL^4}{8EI} + \frac{wL^4}{6EI}(L) \quad \text{(Case 2)} \]

\[ y_4 + y_5 = \frac{\left(\frac{wL^2}{2}\right)L^2}{2EI} + \theta_x L = \frac{\left(\frac{wL^2}{2}\right)L^2}{2EI} + \frac{\left(\frac{wL^2}{2}\right)(L)(L)}{EI} \quad \text{(Case 4)} \]
\[ y_C = y_1 + y_2 + y_3 + y_4 + y_5 \]

\[ y_C = -\frac{w(2L)^4}{8EI} + \frac{wL^4}{8EI} + \frac{wL^2(L)}{6EI} + \left(\frac{wL^2}{2}\right)L^2 + \left(\frac{wL^2}{2}\right)\left(L^2\right) \]

\[ y_C = -\frac{23}{24} \frac{wL^4}{EI} \]

Same problem using Singularity functions:

![Diagram](image)

\[ M_a = \frac{wL^2}{2} \]

\[ M = \frac{wL^2}{2} \]
\[ \sum M_{cur} = M + wL^2 - wLx + \frac{wL^2}{2} + \frac{w}{2} (x - L)^2 = 0 \]

\[ M = -wL^2 + wLx - \frac{wL^2}{2} - \frac{w}{2} (x - L)^2 \]

using Singularity functions

\[ EI \frac{d^2 y}{dx^2} = M = -wL^2 + wL(x - 0) - \frac{wL^2}{2} (x - L)^0 - \frac{w}{2} (x - L)^2 \]

\[ EI \frac{dy}{dx} = -wL^2 x + \frac{wL^2}{2} (x - 0)^2 - \frac{wL^2}{2} (x - L)^1 - \frac{w}{6} (x - L)^3 + C_1 \]

\[ EI \ y = -\frac{wL^2 x^2}{2} + \frac{wL}{6} (x - 0)^3 - \frac{wL^2}{4} (x - L)^2 - \frac{w}{24} (x - L)^4 + C_1 x + C_2 \]

at right end (C), \( x = 2L \), with \( C_1 = C_2 = 0 \),

\[ y_C = y(2L) = -\frac{wL^2 (2L)^2}{2} + \frac{wL (2L)^3}{6} - \frac{wL^2 (L)}{4} - \frac{w(L)^4}{24} \]

\[ y_C = -\frac{23}{24} \frac{wL^4}{EI} \]
Use of superposition for statically indeterminate beams

Example:

FBD:

3 unknown reactions, only 2 available statics equations. The one additional equation can be found by considering deflection at B (i.e., treat $R_B$ as redundant reaction)

Superposition: $y_B = (y_B)_w + (y_B)_{R_B} = 0$

where

(Case 2)

(Case 1)
\[ y_B = \frac{wL^4}{8EI} + \frac{R_b L^3}{3EI} = 0 \quad \therefore R_b = \frac{3}{8} wL \]

Statics: \[ + \uparrow \sum F_y = R_d + R_b - wL = 0 \]
\[ R_d + \frac{3}{8} wL - wL = 0, \quad \therefore R_d = \frac{5}{8} wL \]

\[ + \uparrow \sum M_A = M_A - \frac{wL^2}{2} + R_b L = 0 \]
\[ M_A = \frac{wL^2}{2} - \left( \frac{3}{8} wL \right) (L) = \frac{wL^2}{8} \]

Alternatively, we could treat \( M_A \) as redundant and use:
\[ \theta_A = (\theta_A)_w + (\theta_A)_{M_A} = 0 \]

where
\[ (\theta_A)_w = -\frac{wL^3}{24 EI} \]
\[ (\theta_A)_{M_A} = \frac{M_A L}{3EI} \]

\[ \therefore -\frac{wL^3}{24 EI} + \frac{M_A L}{3EI} = 0 \quad \therefore M_A = \frac{wL^2}{8} \]
From Statics:

\[ R_A = \frac{5}{8} wL \]
\[ R_B = \frac{3}{8} wL \]
as before

Thus, in statically indeterminate beams, the extra equations we need can always be found by considering the boundary conditions on deflections and/or slopes, because every redundant reaction also generates a boundary condition on deflection or slope.

---

Find reactions at A, B, C after loading, assuming contact is made at B after loading.

Superposition: \[ y_B = (y_B)_w + (y_B)_R_B = -1.44\text{"} \]
where

\[(y_B)_w = -\frac{5wL^4}{384 EI}\]

(Case 7)

\[(y_B)_{BA} = -\frac{R_BL^3}{48 EI}\]

(Case 6)

\[y_B = -\frac{5wL^4}{384 EI} + \frac{R_BL^3}{48 EI} = -1.44''\]

\[-\frac{5(800)(20)^4(12)^3}{384 \left(1.6 \times 10^6\right)(500)} + \frac{R_B(20)^3(12)^3}{48 \left(1.6 \times 10^6\right)(500)} = -1.44\]

\[\therefore R_B = 6,000 \text{ lb}\]

\[+ \uparrow \sum F_Y = R_A + R_B + R_C - wL = 0\]

Symmetry : \[R_A = R_C\]

\[\therefore R_A = R_C = 5000 \text{ lb}\]
Find the tension in the aluminum rod.

Superposition:

\[(y_B)_w + (y_B)_w = -\delta_{rod} = -\frac{R_BL_{rod}}{A_{rod}E_{rod}}\]

where

\[(y_B)_w = -\frac{wL^4}{8EI}\]  

(Case 2)

\[(y_B)_w = \frac{R_BL^3}{3EI}\]  

(Case 1)
\[
\begin{align*}
(y_B)_w + (y_B)_R & = -\frac{R_B L_{rod}}{A_{rod} E_{rod}} \\
- \frac{w L^4}{8EI} + \frac{R_B L^3}{3EI} &= -\frac{R_B L_{rod}}{A_{rod} E_{rod}} \\
- \frac{(2000 \times 10^{9}) (3 \times 10^{-9})}{8(8 \times 10^{-1}) (6 \times 10^{-7})} + \frac{R_B (3 \times 10^{9}) (10^{-9})}{3(8 \times 10^{-1}) (6 \times 10^{-7})} &= -\frac{R_B (100 \times 72 \times 10^{-9})}{(100 \times 72 \times 10^{-9})} \\
- 42.19 + 18.75 \times 10^{-3} R_B &= 1.389 \times 10^{-3} R_B \\
\therefore R_B &= 2095 \text{ N}
\end{align*}
\]

**Stress Analysis of Statically Indeterminate beam**

Find the maximum normal stress and the maximum shear stress in the beam shown below.

![Beam diagram with w=16 kN/m, 2 m, 4 m, and A, B, C labels]
Beam X-Section:

\[
\begin{align*}
&w = 16 \text{ KN/m} \\
&R_A, R_B, R_C
\end{align*}
\]

Superposition:

From integration:

\[
(y_B)_w = -\frac{w}{24EI} \left( x^4 - 2Lx^3 + L^3x \right)
\]

\[
= -\frac{w}{24EI} \left( 2^4 - 2(6)(2)^3 + (6)^3(2) \right)
\]

\[
= -\frac{14.67w}{EI}
\]
From integration:

for \( x = a \)

\[
(y_B)_{Ra} = \frac{R_a a^2 b^2}{3EI} = \frac{R_a (2)^2 (4)^2}{3EI(6)} = \frac{3.55R_a}{EI}
\]

Compatibility:

\[
(y_B)_a + (y_B)_{Ra} = 0
\]

\[
-14.67w + \frac{3.55R_a}{EI} = 0
\]

\[
\therefore R_a = \frac{14.67w}{3.55} = 4.132 \text{ w}
\]

\[
\therefore R_a = 4.132 (6) = 66 \text{ kN}
\]

\[
\sum F_y = R_A + R_B + R_C - wL = 0
\]

\[
R_A + 66 + R_C - (16)(6) = 0
\]

\[
R_A = 30 - R_C
\]

\[
\sum M_B = 2R_A + 16(6) - 4R_C = 0
\]

\[
2(30 - R_C) + 96 - 4R_C = 0
\]

\[
\therefore R_C = 26 \text{ KN}
\]

\[
R_A = 30 - 26 = 4 \text{ KN}
\]
$w = 16 \text{ kN/m}$

$V(kN) = 4 \text{ kN}$

$M(kN\cdot m) = 4 \text{ kN} \cdot 4 \text{ m} - 4 \text{ kN} \cdot 2 \text{ m} = 24 \text{ kN} \cdot \text{m}$

$\frac{dV}{dX} = -w$

$\frac{dM}{dX} = V$

$\overline{C} = \frac{280 \left( \frac{200}{280} \right) (100) - 160 (140)(130)}{200 (200) - 160 (140)} = 80 \text{ mm}$

$I = 2 \left[ \frac{60(140)^3}{12} + 60(140)(50)^2 \right] + \frac{280(60)^3}{12} + 280(60)(50)^2$

$I = 116.48 \times 10^6 \text{ mm}^4$
The steel beam (E = 30 × 10^6 psi, I = 146 in^4) on the left is found to sag excessively at the mid span. It is proposed that a spring, k, be placed under the beam at mid span in order to limit the mid span deflection to 2 inches. Assuming that the spring is unstressed before loading, find the spring stiffness, k, that would be required.

The steel beam (E = 30 × 10^6 psi, I = 146 in^4) on the left is found to sag excessively at the mid span. It is proposed that a spring, k, be placed under the beam at mid span in order to limit the mid span deflection to 2 inches. Assuming that the spring is unstressed before loading, find the spring stiffness, k, that would be required.

\[ \sigma_{x_{\text{max}}} = -\frac{My}{I} = -\frac{(-24 \times 10^3)(120 \times 10^{-3})}{(116.48 \times 10^6)(10^{-12})} \]
\[ \sigma_{x_{\text{max}}} = 24.72 \times 10^6 \frac{N}{m^2} = 14.72 \text{ MPa} \]

@ Neutral axis

\[ \tau_{y_{\text{max}}} = \frac{VQ}{It} = \frac{(38 \times 10^3)(2)(120)(60)(60)}{(116.48 \times 10^6)(120)(10^{-6})} \]
\[ \tau_{y_{\text{max}}} = 2.35 \times 10^6 \frac{N}{m^2} = 2.35 \text{ MPa} \]
For beam with spring, the mid span deflection is
\[(y_B)_w + (y_B)_{R_s} = \frac{R_B}{k} = -2''\]

where
\[(y_B)_w = -\frac{5wL^4}{384EI}\]
\[(y_B)_{R_s} = \frac{R_BL^3}{48EI}\]
\[\therefore \frac{5wL^4}{384EI} + \frac{R_BL^3}{48EI} = -2''\]

\[\frac{-5(1000)(30)^4(1728)}{384\left(30 \times 10^6\right)(146)} + \frac{R_B(20)^3(1728)}{48\left(30 \times 10^6\right)(146)} = -2''\]

\[\therefore R_B = 9,738 \text{ lb}\]

and \[k = \frac{R_B}{y_B} = \frac{9,738}{2} = 4,869 \text{ lb/in}\]
Determine the reactions at the clamped ends of the two cantilever beams, which are hinged together at B as shown.

Each beam is statically indeterminate, so deformation equations must be used to supplement statics equation

Geometric compatibility: \((y_1)_B = (y_2)_B\)

where

\[(y_1)_B = -\frac{R_B L^3}{3EI}\]
(y_2)_a = -\frac{P\left(\frac{3L}{4}\right)^3}{3EI} - \theta_c\left(\frac{L}{4}\right) + \frac{R_y L^3}{3EI} - \frac{P\left(\frac{3L}{4}\right)^2}{2EI} \left(\frac{L}{4}\right) + \frac{R_y L^3}{3EI}.

substituting in (y_1)_a = (y_2)_a and find that \( R_B = \frac{81}{256}P \)

Then applying statics equations to above FBD's, find the remaining reactions:

\[ \begin{array}{ccc}
81\frac{PL}{256} & 81\frac{P}{256} & 111\frac{PL}{192} \\
81\frac{P}{256} & 81\frac{P}{256} & 175\frac{P}{256} \\
\end{array} \]