Stresses

Normal stress
(Note this is average value)

\[ \sigma = \frac{P}{A} \]

Example: Normal stress in member \( AB \)

\[ \sigma_{AB} = \frac{F_{AB}}{A_{AB}} \]

Normal Stress at a point

\[ \sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \]

\[ P = \int dF = \int \sigma dA \]
Stress concentrations at points of load application

Stress not uniform due to M

Units of Stress

**SI units:**

- Pascal $\sigma = \frac{N}{m^2}$
- Kilo Pascal $= kPa = 10^3 \frac{N}{m^2}$
- Mega Pascal $= MPa = 10^6 \frac{N}{m^2}$

**U.S. units:**

- $\frac{lb}{in^2} = \text{Pounds per square inch} = \text{psi}$

**Conversion:**

- $1 \text{ psi} = 6.895 \text{ kPa}$
- $1 \text{ ksi} = 6.895 \text{ Mpa}$
Shear Stress

$F_j = \text{Shear force on glue joint}$

Average shear stress $\tau_{\text{joint}} = \frac{F_j}{A_j}$

Stress at a point $\tau = \lim_{A \to 0} \frac{\Delta F}{\Delta A}$

Example: pin at A in truss member

$F_{AB} \quad \frac{F_{AB}}{2} \quad \frac{F_{AB}}{2} \quad \frac{F_{AB}}{2}$

$F_{\text{pin}} = \frac{F_{AB}}{2} = \text{shear force}$

$\tau_{\text{pin}} = \frac{F_{\text{pin}}}{A_{\text{pin}}} = \frac{F_{AB}}{2A_{\text{pin}}}$
Stress on Oblique Plane under Axial Loading

Two directions required to specify stress component, one for orientation of face and another for force acting on face

Perpendicular plane:

\[
\sigma_{xx} = \frac{P_{xx}}{A_x} \text{ (uniform)}
\]
Parallel plane:

\[ \tau_{yx} = \frac{P_{yx}}{A_y} \]

Oblique plane:

\[ \sigma_{xx} = \frac{P_{xx}}{A_x} \]

Note:

\[ A_{x'} = \frac{A_X}{\cos \theta} \]
Force Normal to Oblique Plane  = \( P_{xx} \cos \theta = P_{xx'} \)

Force Parallel to Oblique Plane  = \( P_{xx} \sin \theta = P_{xy} \)

**Normal Stress**

\[
\sigma_{x'} = \frac{P_{xx}}{A_x} = \sigma_{x''}
\]

or

\[
\sigma_{x'} = \left( P_{xx} \cos \theta \right) \left( \frac{\cos \theta}{A_x} \right) = \frac{P_{xx}}{A_x} \cos^2 \theta = \sigma_x \cos^2 \theta
\]

**Shear Stress**

\[
\tau_{xy'} = \frac{P_{yy}}{A_y} = P_{xx} \sin \theta \left( \frac{\cos \theta}{A_x} \right)
\]

\[
\tau_{xy'} = \sigma_x \sin \theta \cos \theta
\]
Variation of normal stress and shear stress with angle $\theta$

\[
\sigma_{\text{max}} = \sigma_X = \frac{P_{XX}}{A_X}
\]
\[$@ \theta = 0^\circ\]

\[
\tau_{\text{max}} = \frac{\sigma_X}{2} = \frac{P_{XX}}{2A_X}
\]
\[$@ \theta = 45^\circ\]

\[
\sigma_{\text{min}} = 0 \quad @ \theta = 90^\circ
\]
\[
\tau_{\text{min}} = 0^\circ, 90^\circ
\]

Importance of Normal Stresses and Shear Stresses in Material Failure

1. Brittle materials fail by maximum tensile normal stress
2. Ductile materials fail by maximum shear stress at $45^0$

Brittle:

\[
\]

Ductile:

\[
\]
Actual-cup-cone fracture surface for ductile material

The bar of varying cross-sectional dimensions is attached to a rigid support by a weld joint and is centrically loaded at several locations as shown below. Assume that the bar has a rectangular cross-section with unit thickness in the direction perpendicular to the page along its entire length. Determine the stresses acting on each of the indicated planes A, B, C and D, where each plane is assumed to be perpendicular to the page. Answers should be expressed in terms of the given parameters.
Plane A: Free Body Diagram

\[ V_A \]

\[ \sum F_X = 2P - 2V_A + P - 2P = 0 \]

\[ \therefore V_A = \frac{P}{2} \]

\[ \tau_A = \frac{V_A}{A_A} = \frac{P/2}{(a)(b)} = \frac{P}{2A} \]

Solution:

Plane B: Free Body Diagram

\[ P_B \]

\[ \sum F_X = P_B + P - 2P = 0 \]

\[ \therefore P_B = P \]

\[ \sigma_B = \frac{P_B}{A_B} = \frac{P}{(b)(1)} = \frac{P}{b} \]
Plane C: Free Body Diagram

Solution:
\[ \sum F_X = P_C - 2P = 0 \quad \therefore P_C = 2P \]

On Plane C:
\[ \sigma_x = \frac{2P}{c(1)} = \frac{2P}{c} \]
\[ \therefore \sigma'_x = \sigma_x \cos^2 \theta = \frac{2P}{c} \cos^2 45^\circ = \frac{P}{c} \]
\[ \tau_{xy} = \sigma_x \sin \theta \cos \theta = \frac{2P}{c} \sin 45^\circ \cos 45^\circ = \frac{P}{c} \]

Plane D: Free Body Diagram

Solution:
\[ \sum F_X = P_D = 0 \]
\[ \therefore \sigma_D = 0 \]
Allowable Stresses:

Shear stress along grain
\[ \tau_{allow} = 185 \text{ psi} \]

Normal stress perpendicular to grain
\[ \sigma_{allow} = 365 \text{ psi} \]

Find the maximum value of \( P \) so that \( \tau_{allow} \) or \( \sigma_{allow} \) are not exceeded

Normal Stress:

\[ \sigma_{X'} = \sigma_x \cos^2 \theta = \frac{P}{A} \cos^2 \theta \]

\[ 365 = \frac{P}{4(4)} \cos^2 30^\circ \]

\[ P = 7787 \text{ lb} \]
Shear Stress:
\[ \tau_{xy} = \sigma_x \sin \theta \cos \theta = \frac{P}{A} \sin \theta \cos \theta \]

\[ 185 = \frac{P}{4(4)} \sin 30 \degree \cos 30 \degree \]

\[ P = 6836 \text{ lb} \quad \text{and the shear stress governs} \]

Find the allowable load P if the maximum allowable shear stress along the glue joint is 100 psi and the maximum allowable tensile normal stress across the glue joint is 300 psi
\[
\sum F_Y = -P + \frac{4}{5} F_{AB} = 0 \\
\therefore F_{AB} = \frac{5}{4} P \\
\sum F_X = -F_{BC} + \frac{3}{5} F_{AB} = -F_{BC} + \frac{3}{5} \left( \frac{5}{4} P \right) = 0 \\
\therefore F_{BC} = \frac{3}{4} P
\]
Shear stress:

\[ \tau_{xy} = \sigma_x \sin \theta \cos \theta = \frac{3}{4} P \frac{\sin 60^\circ \cos 60^\circ}{(2)(4)} \]

\[ = 100 \text{ psi} \]

\[ \therefore P = 2,463 \text{ lb} \]

therefore failure by shear governs and \( P = 2,463 \text{ lb} \) is the maximum allowable load.

Stress at a point: consider an imaginary cut perpendicular to the X- axis as shown below.
Normal stress on cut:
\[
\sigma_{xx} = \lim_{\Delta A_x \to 0} \left( \frac{\Delta F_x}{\Delta A_x} \right)
\]

Shear stresses on cut:
\[
\tau_{xy} = \lim_{\Delta A_x \to 0} \left( \frac{\Delta F_y}{\Delta A_x} \right)
\]
\[
\tau_{xz} = \lim_{\Delta A_x \to 0} \left( \frac{\Delta F_z}{\Delta A_x} \right)
\]

Similarly, we can find three components of stress acting on the Y and Z faces
\[
\sigma_{yy} = \lim_{\Delta A_y \to 0} \left( \frac{\Delta F_y}{\Delta A_y} \right) \quad \sigma_{zz} = \lim_{\Delta A_z \to 0} \left( \frac{\Delta F_z}{\Delta A_z} \right)
\]
\[
\tau_{yx} = \lim_{\Delta A_y \to 0} \left( \frac{\Delta F_x}{\Delta A_y} \right) \quad \tau_{zx} = \lim_{\Delta A_z \to 0} \left( \frac{\Delta F_x}{\Delta A_z} \right)
\]
\[
\tau_{yz} = \lim_{\Delta A_y \to 0} \left( \frac{\Delta F_z}{\Delta A_y} \right) \quad \tau_{zy} = \lim_{\Delta A_z \to 0} \left( \frac{\Delta F_y}{\Delta A_z} \right)
\]
Due to moment equilibrium:

\[
\tau_{XY} = \tau_{YX} \\
\tau_{XZ} = \tau_{ZX} \\
\tau_{YZ} = \tau_{ZY}
\]

(Will show later)

Strain

Normal strain under Axial Loading

\[
\text{Normal strain } = \varepsilon = \frac{\delta}{L} \quad \text{(for constant uniform stress)}
\]
Normal strain at a point

\[ \varepsilon = \lim_{\Delta X \to 0} \left( \frac{\Delta \delta}{\Delta X} \right) \]  
(for varying stress)

\[ X + \delta \quad \Delta X + \delta \]

Strains in structural materials are generally very small

\[ \varepsilon \]  
on order of \( 10^{-6} \) \( \text{in} \) or \( \text{mm} \)

\[ \therefore \]  
“microstrain” = \( 10^{-6} \) \( \text{in} \) or \( \text{mm} \)

more useful to work with \( \sigma \) and \( \varepsilon \) rather than \( P \) & \( \delta \) . \( \sigma \) & \( \varepsilon \) apply in any situation, whereas \( \delta \) & \( P \) are specific to a given structural element.

\[ \sigma - \varepsilon \]  
curves

Brittle material – Linear curve to Fracture (e.g., glass, ceramics)

\( \sigma_b = \) breaking stress
\( \sigma_u = \) ultimate stress

\( \varepsilon_u = \sigma_b \)

\( \varepsilon_u = \sigma_b \)

Fracture surface \( \perp \) to load
Ductile Materials

Cup-cone failure surface with 45° Shear lips

Example of ductile material-
Low Carbon Steel

\[
E = \frac{\sigma}{\varepsilon}
\]
Example of ductile material - Aluminum Alloy

\[ \sigma' = \frac{\sigma}{\varepsilon} \]

Percent elongation \( = 100 \times \sigma_B = \frac{L_B - L_O}{L_O} \)

For linear Elastic Region Hooke’s law applies

\[ \sigma = E \varepsilon \quad (1-D \text{ only!}) \]

\( E = \text{Young’s modulus} \)

Units: \( \text{ksi} = 10^3 \text{ psi} , 10^6 \text{ psi} \),

\( \text{Mpa} = 10^6 \frac{N}{m^2} , \text{ Gpa} = 10^9 \frac{N}{m^2} \)

see Appendix A, Tables A-17, A-18 (table of properties)

Conversion:

\( 1 \text{ksi} = 6.895 \text{ MPa} , 10^6 \text{ psi} = 6.895 \text{ GPa} \)
Inelastic Behavior - Slip and dislocation movement, plastic deformation

Slip occurs by passage of dislocations along slip planes and not by simultaneous shearing of atomic planes - "rug wrinkle" analogy.

Large force is required to pull rug across floor

Rug can be moved with less force by creating wrinkle and pushing it along from one end to the other
Strain hardening (work hardening)
large scale dislocation movement

Engineering Stress-Strain curve
True Stress-Strain Curve
Plastic region
Linear elastic region

Engineering vs. True Stress and Strain

Engineering
True

\[ \sigma = \frac{P}{A_o} \quad \sigma = \frac{P}{A_i} \]

\[ \varepsilon = \frac{\delta}{L_o} \quad \varepsilon = \int \frac{dL}{L} = \ln \left( \frac{L_f}{L_o} \right) \]

based on original dimensions \( A_o, L_o \)
based on instantaneous dimensions
Shear Strain

Unlike normal strain, which characterizes elongation or contraction under normal stress, shear strain characterizes distortion under shear stress.

\[ \gamma_{xy} = \text{Shear strain (radians)} \]

or shear angle (dimensionless)

\[ \gamma_{xy} = \frac{\tau_{xy}}{L} \]

Rotating deformed shape CW by \( \frac{\gamma_{xy}}{2} \) for convenience,

for small deformations, the average shear strain is

\[ \gamma_{xy} = \tan \gamma_{xy} = \frac{\delta_s}{L} \]

Where \( \delta_s \) shear deformation

For normal form shear strain, need to define shear strain at a point

\[ \gamma_{xy} = \lim_{L \to 0} \left( \frac{\Delta \delta_s}{\Delta L} \right) = \frac{d \delta_s}{dL} \]
Deformations under axial load

\[ L \quad P \]

\[ L + \delta \]

Stress: \( \sigma = \frac{P}{A} \); Strain: \( \varepsilon = \frac{\delta}{L} \)

Hooke’s Law: \( \sigma = E \varepsilon \) (one dimensional)
(for linear elastic region)

Substituting above equations
\[
\sigma = \frac{P}{A} = E \varepsilon = E \frac{\delta}{L}
\]

\[ \therefore \delta = \frac{PL}{AE} \] Elongation (or contraction) of uniaxially loaded element

Example: this equation applies to any 2-force member in a loaded structure

[Diagram showing force and deformation]
Members AB and BC are made of steel \((E = 30 \times 10^6 \text{ psi})\) and each member has a cross sectional area of \(2.5 \text{ in}^2\). Find the stresses and the axial deflections of each member.

\[
\begin{align*}
F_{AB} & \uparrow F_{BC}' \downarrow P = 10,000 \text{ lb} \\
F_{BC} & \downarrow F_{AB}' \uparrow \\
A & \quad B \\
C & \\
4' & \quad 3'
\end{align*}
\]

Statics:
\[
\begin{align*}
\sum F_y & = 0 \\
\Rightarrow F_{BC} - \frac{4}{5} F_{BC} - P & = 0 \\
\therefore F_{BC} & = \frac{5}{4} P = 12,500 \text{ lb} \\

\sum F_x & = 0 \\
\Rightarrow \frac{3}{5} F_{BC} - F_{AB} & = 0 \\
\therefore F_{AB} & = \frac{3}{5} \left( \frac{5}{4} P \right) = 7,500 \text{ lb}
\end{align*}
\]

Deflections:
\[
\delta_{BC} = \frac{F_{BC} L_{BC}}{A_{BC} E_{BC}} = \frac{(12,500 \times \frac{5}{4}) \times 12}{(2.5) \times (30) \times 10^6} = 0.01 \text{ in} \quad \text{(contraction)}
\]
\[ \delta_{AB} = \frac{F_{AB} L_{AB}}{A_{AB} E_{AB}} = \frac{(7,500)(3)(12)}{(2.5)(30) \times 10^6} = 0.0036 \text{ in (extension)} \]

Stresses:
\[ \sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{12,500}{2.5} = 5,000 \text{ psi (C)} \]
\[ \sigma_{AB} = \frac{F_{AB}}{A_{AB}} = \frac{7,500}{2.5} = 3,000 \text{ psi (T)} \]

A concrete column (E = 20 GPa) having a variable cross section is subjected to several loads as shown. Find the Stresses and deflections of elements AB, BC and CD and the total vertical deflection at A.

<table>
<thead>
<tr>
<th>Element</th>
<th>Area (m²)</th>
<th>Length (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>BC</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>CD</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ \text{Diagram showing load distribution and cross sections of AB, BC, CD.} \]
AB: 

\[ \sum F_y = 10 - F_{AB} = 0, \ F_{AB} = 10 \text{ KN} \ (T) \]

\[ \sigma_{AB} = \frac{F_{AB}}{A_{AB}} = \frac{10}{6} = 1.67 \text{ KN} / \text{m}^2 \ (T) \]

\[ \delta_{AB} = \frac{F_{AB}L_{AB}}{A_{AB}E_{AB}} = \frac{10(1)(10^3)}{(6)(20 \times 10^9)} = 8.33 \times 10^{-8} \text{ m} \]

(elongation)

BC: 

\[ \sum F_y = 10 - 2(6) + F_{BC} = 0, \ F_{BC} = 2 \text{ KN} \ (C) \]

\[ \sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{2}{8} = 0.25 \text{ KN} / \text{m}^2 \ (C) \]

\[ \delta_{BC} = \frac{F_{BC}L_{BC}}{A_{BC}E_{BC}} = \frac{2(1)(10^3)}{8(20 \times 10^9)} = 1.25 \times 10^{-8} \text{ m} \]

(contraction)

CD: 

\[ \sum F_y = 10 - 2(6) - 2(2) + F_{CD} = 0, \ F_{CD} = 6 \text{ KN} \ (C) \]

\[ \sigma_{CD} = \frac{F_{CD}}{A_{CD}} = \frac{6}{10} = 0.6 \text{ KN} / \text{m}^2 \ (C) \]

\[ \delta_{CD} = \frac{F_{CD}L_{CD}}{A_{CD}E_{CD}} = \frac{6(1)(10^3)}{10(20 \times 10^9)} = 3 \times 10^{-8} \text{ m} \]

(contraction)

\[ \delta_{A} = \delta_{AB} + \delta_{BC} + \delta_{CD} = (8.33 - 1.25 - 3) \times 10^{-8} = \]

\[ \delta_{A} = 4.08 \times 10^{-8} \text{ m} \]

(↑)
Example of internal force diagram showing distribution of internal forces along length of structure

Thermal Strain and deflection:
Isotropic materials expand or contract uniformly in all directions under temperature change $\Delta T$.

For bar, the thermal strain is $\varepsilon_T = \alpha (\Delta T)$
Where $\alpha$ = coefficient of thermal expansion (CTE)
and deflection is $\delta_T = \varepsilon_T L = \alpha L (\Delta T)$
So the total strain in bar along axis is $\varepsilon = \frac{\sigma}{E} + \alpha (\Delta T)$

$\Delta \varepsilon = 0$,
$\Delta \varepsilon = \Delta \varepsilon_T = \alpha \Delta T$

and total deflection is

$\delta = \frac{PL}{AE} + \alpha L (\Delta T)$

Mechanical part
Thermal part
Example: free expansion with no applied loads and temperature change $\Delta T$.

Strain: $\varepsilon = 0 + \alpha (\Delta T)$, $P = 0$

Stress: $\sigma = \frac{P}{A} = 0$

Example: Restrained

![Diagram of a restrained system with rigid walls and temperature change $\Delta T$.]

Total strain: $\varepsilon = \frac{P}{AE} + \alpha (\Delta T) = 0$

Total deflection: $\delta = \frac{PL}{AE} + \alpha L (\Delta T) = 0$

∴ Thermal stress is $\sigma = \frac{P}{A} = -E \alpha (\Delta T)$
Statically Indeterminate Problems
(or, know when to give up with statics!)
When the number of unknown reactions is greater than the number of independent, nontrivial static equilibrium equations, the structure is statically indeterminate and the unknown reactions cannot be determined from statics alone. Additional equations must be generated by considering deformations.
Example: Concrete post under compressive load $P$

\[ P \quad \therefore \text{if } \Delta T > 0, \text{ Stress is compressive} \]
\[ P \quad \therefore \text{if } \Delta T < 0, \text{ Stress is tensile} \]

\[ \therefore \text{ Thermal stresses only develop when thermal deformation is restrained. If no restraint, we get free expansion and thermal strain } \epsilon = \alpha (\Delta T) \]
For rigid block:
\[ + \sum F_y = -P + F_c = 0 \therefore F_c = P \]

And since \( F_c \) can be determined from statics, the structure is **statically determinate**.

However, if we attach a steel reinforcement bar to the concrete post, we make the structure **statically indeterminate**.

Since there are two unknown reactions:

\[ \begin{align*}
\sum F_x &= 0 \\
\sum F_y &= 0
\end{align*} \]
One additional equation is needed to supplement the statics equation, and that equation can be generated by considering the deformations.

Geometric compatibility: \( \delta_s = \delta_c \)
where \( \delta_s \) = deformation of steel
\( \delta_c \) = deformation of concrete

Force-Deformation:
\[
\delta_s = \frac{F_s L_s}{A_s E_s} \quad \text{and} \quad \delta_c = \frac{F_c L_c}{A_c E_c}
\]
Combining these equations, we get
\[
\frac{F_s L_s}{A_s E_s} = \frac{F_c L_c}{A_c E_c}
\]

Since \( L_s = L_c \)
\[
F_s = \left( \frac{A_s E_s}{A_c E_c} \right) F_c
\]

Since we know the cross sectional areas \( A_s \) and \( A_c \) and The moduli of elasticity \( E_s \) and \( E_c \), we can solve this equation simultaneously with the statics equation.
\[
\sum F_y = -P + F_c + F_s = 0
\]
And determine the unknown reactions \( F_c \) and \( F_s \).
Example: if $P = 42,000$ lb determine stresses in bars

2 unknown reactions $F_S$ & $F_B$
1 independent non-trivial
statics equation $\sum F_y = 0$

Geometry: $\delta_S = \delta_B$ at the block

Force-deflection: $\delta_S = \frac{F_S L_S}{A_S E_S}$; $\delta_B = \frac{F_B L_B}{A_B E_B}$

\[
\therefore \frac{F_S (6)}{(1)(30 \times 10^6)} = \frac{F_C (2)(12)}{(4)(15 \times 10^6)}
\]

$6 F_S = F_B$

Equilibrium:
Superposition: Another method for solving statically indeterminate problems. For example, if we remove the support at A and analyze the structure as shown below, the geometric compatibility equation is

\[ (\delta_A)_P - (\delta_B)_{F_S} = 0 \]
First consider $P$ alone:

$$
(\delta_A)_P = \frac{PL_B}{A_B E_B} + 0 \downarrow \\
(\text{note that only the bar B deforms here})
$$

Now consider $F_S$ alone:

$$
(\delta_A)_{F_S} = \frac{F_SL_S}{A_S E_S} + \frac{F_SL_B}{A_B E_B} \uparrow \\
(\text{note that the bars B and S both deform now under the same load } F_S)
$$

Now, 

$$
(\delta_A)_P - (\delta_B)_{F_S} = 0 \quad \text{since point A is fixed} \\
\frac{PL_B}{A_B E_B} - \left( \frac{F_SL_S}{A_S E_S} + \frac{F_SL_B}{A_B E_B} \right) = 0
$$
The rod of length $L$, modulus of elasticity $E$ and diameter $D$ is fixed on one end and free on the other end before the load is applied. Before the load is applied, there is a gap of width $A$ between the free end of the rod and the wall. After the axial load $P$ is applied, the free end of the rod contacts the wall as shown.

Answer each of the questions below in terms of the given parameters.

\[
\frac{42,000}{4(15)} (2) - F_s \left( \frac{6}{1(30)} + \frac{2}{4(15)} \right) = 0
\]

\[
F_s = 6,000 \text{ lb}
\]

Then from statics, $F_b = 36,000 \text{ lb}$, etc. as before
Draw the free-body diagram of the rod after the load is applied

\[
\begin{array}{c}
Y \\
\hline
X \\
\hline
\end{array} \quad \begin{array}{c}
F_A \\
F_B \\
\end{array} \quad \begin{array}{c}
F_P \\
\end{array}
\]

Write the static equilibrium equation for the above free body diagram

\[
\sum F_x = P - F_A - F_B = 0
\]

\[
\therefore F_A = P - F_B
\]

Write the geometric compatibility equation

\[
\delta = \Delta
\]

Write the force-deformation equation

\[
\delta = \frac{F_A L}{AE} = \left(\frac{P - F_B}{L}\right) E = \Delta
\]

\[
\frac{F_A}{A} = \left(\frac{\pi D^2}{4}\right) \frac{\Delta E}{L}
\]

Determine the stress in the rod after the load is applied.

\[
\sigma = \frac{F_A}{A} = \frac{P - F_B}{L} = \frac{\Delta E}{L}
\]

Reaction Force: \[ F_B = P - F_A = P - \frac{\Delta AE}{L} \]
Example:

Find the stress in $A_2$

$$(\delta_A)_P = \frac{Pb}{A_2E}$$
\( (\delta_A)_{R_A} = \frac{R_A b}{A_1 E} + \frac{2 R_A a}{A_1 E} \)

\( (\delta_A)_{total} = (\delta_A)_p + (\delta_A)_{R_A} = 0 \)

\[
\frac{P A_2 b}{A_1 E} = R_A \left( \frac{2 a}{A_1 E} + \frac{b}{A_2 E} \right)
\]

\[
R_A = \frac{P b}{A_2 \left( \frac{2 a}{A_1} + \frac{b}{A_2} \right)}
\]

\[\sum F_X = P - R_A - P_2 = 0 \quad \therefore \quad P_2 = P - R_A\]

\[
\therefore \sigma_2 = \frac{P_2}{A_2} = \frac{P}{A_2} \left[ 1 - \frac{b}{A_2 \left( \frac{2 a}{A_1} + \frac{b}{A_2} \right)} \right]
\]
A thin-walled hemispherical pressure vessel is attached to a flat place with a flange and eight (8) bolts, as shown below. The allowable stress in each bolt is $\sigma_{\text{max}}$, the bolt diameter is $d$, the stresses length of each bolt is $L$, the mean diameter of the pressure vessel is $D$, and the modulus of elasticity of each bolt is $E$. The bolts are initially unstressed and the vessel is then pressurized to pressure $P$. Answer each of the questions below in terms of the given parameters.

**Draw the free-body diagram of the pressure vessel**

**Write the static equilibrium equation for the above free body diagram**

$$
\sum F_y = 0 \quad \therefore \quad P \frac{\pi D^2}{4} = 8 F_B
$$
Write the geometric compatibility equation:

Assuming that bolts are loaded in parallel:

\[ \delta_{B_1} = \delta_{B_2} = \delta_{B_3} = \delta_{B_4} = \delta_{B_5} = \delta_{B_6} = \delta_{B_7} = \delta_{B_8} \]

(all bolts have same deflection)

Write the force-deformation equation

\[
\delta_{B} = \frac{F_{B} L}{AE} = \sigma_{\text{max}} \frac{L}{E}
\]

\[ \therefore F_{B} = A \sigma_{\text{max}} = \frac{\pi d^2}{4} \sigma_{\text{max}} \]

Determine the allowable pressure, \( P_{\text{max}} \), in the vessel

\[
P_{\text{max}} \frac{\pi D^2}{4} = 8 F_{B} = 8 \frac{\pi d^2}{4} \sigma_{\text{max}}
\]

\[ \therefore P_{\text{max}} = 8 \left( \frac{d}{D} \right)^2 \sigma_{\text{max}} \]

Determine the allowable elongation, \( \delta_{\text{max}} \), of each bolt

\[
\delta_{\text{max}} = \frac{F_{B} L}{AE} = \sigma_{\text{max}} \frac{L}{E}
\]
Find forces in members and deflection of pin C:

\[ K_1 = \left( \frac{AE}{L} \right)_1 \text{ etc} \]

\[ K_1 = K_3 (\text{symmetry}) \]

Equilibrium:

\[ \sum F_x = F_1 \sin \theta - F_3 \sin \theta = 0 \]

\[ \therefore F_1 = F_3 \]

\[ \sum F_y = F_1 \cos \theta + F_2 + F_3 \cos \theta - P = 0 \]

\[ (1) \quad P = F_2 + 2F_1 \cos \theta \]

Statically indeterminate
Force -Deflection:

\[
\begin{align*}
F_1 &= K_1 \delta_1 = F_3 \quad \{2\} \\
F_2 &= K_2 \delta_2
\end{align*}
\]

Geometric Compatibility:

\[
\delta_1 = \delta_2 \cos \theta \quad (3)
\]

\[
\therefore \quad \frac{F_1}{K_1} = \frac{F_2}{K_2} \cos \theta \quad (2)+(3)
\]

from this

\[
F_1 = \frac{F_2 K_1}{K_2} \cos \theta
\]

substituting (1)

\[
P = F_2 + 2 \frac{F_2 K_1}{K_2} \cos^2 \theta
\]

\[
F_2 = \frac{P}{1 + 2 \frac{K_1}{K_2} \cos^2 \theta}
\]

then

\[
F_1 = \frac{F_2 K_1}{K_2} \cos \theta
\]

\[
\delta_2 = \text{deflection at D} = \frac{F_2}{K_2} = \frac{P}{K_2 + 2 K_1 \cos^2 \theta}
\]
Find the equivalent stiffness for the two spring arrangements shown below

Parallel arrangement

For the parallel arrangement, the free body diagrams are as shown below

Equilibrium equation:
\[ \sum F_y = -P + F_1 + F_2 = 0 \]
\[ \therefore P = F_1 + F_2 \]

Geometric Compatibility:
assuming the spring supports remain parallel: \( \delta = \delta_1 = \delta_2 \)

Force-deformation:
\[ F_1 = K_1 \delta_1 = K_1 \delta \]
\[ F_2 = K_2 \delta_2 = K_2 \delta \]
Combining the above equations, we get

\[ P = (K_1 + K_2)\delta \]

So the equivalent stiffness of two parallel springs is

\[ K_{eq} = \frac{P}{\delta} = K_1 + K_2 \]

For a system of n springs in parallel,

\[ K_{eq} = \sum_{i=1}^{n} K_i \]

For the series arrangement, the free body diagrams are as shown below

It is clear that static equilibrium requires that

\[ P = F_1 = F_2 \]

Geometric Compatibility:
the total deformation is \[ \delta = \delta_1 + \delta_2 \]

Force-deformation:
\[ F_1 = K_1\delta_1, \quad F_2 = K_2\delta_2 \]
Combining the above equations, we get

\[ \delta = \frac{F_1}{K_1} + \frac{F_2}{K_2} = P \left( \frac{1}{K_1} + \frac{1}{K_2} \right) \]

So the equivalent stiffness of two series springs is then

\[ K_{eq} = \frac{P}{\delta} = \frac{1}{\frac{1}{K_1} + \frac{1}{K_2}} \]

For a system of \( n \) springs in series,

\[ K_{eq} = \frac{1}{\sum_{i=1}^{n} \frac{1}{K_1}} \]

Note that for a uniform bar loaded with an axial load, the stiffness \( K \) is found as shown below:

\[ \delta = \frac{PL}{AE} \]

\[ \therefore K = \frac{P}{\delta} = \frac{AE}{L} \]
Series and parallel arrangement of axially loaded members

\[ \delta = \frac{PL}{AE} \]

**Equivalent spring**

Spring stiffness, \( K \)

\[ K = \frac{P}{\delta} = \frac{AE}{L} \]

**Series arrangement**

Forces: \( P_1 = P_2 = P_{\text{total}} \)

Deformations:

\[ \delta_{\text{total}} = \delta_1 + \delta_2 \]

Equivalent stiffness:

\[ K_{\text{eq}} = \frac{P_{\text{total}}}{\delta_{\text{total}}} = \frac{P_{\text{total}}}{\delta_1 + \delta_2} \]

or

\[ \frac{1}{K_{\text{eq}}} = \frac{\delta_1}{P_1} + \frac{\delta_2}{P_2} = \frac{1}{K_1} + \frac{1}{K_2} \]
Parallel arrangement

Forces: \( P_{\text{total}} = P_1 + P_2 \)

Deformations:
\( \delta_{\text{total}} = \delta_1 = \delta_2 \)

Equivalent stiffness:
\[
K_{eq} = \frac{P_{\text{total}}}{\delta_{\text{total}}} = \frac{P_1 + P_2}{\delta_{\text{total}}}
\]
\[
K_{eq} = \frac{P_1}{\delta_1} + \frac{P_2}{\delta_2} = K_1 + K_2
\]
Designing Against Failure
The design of structural and machine elements against failure involves the selection of materials, shapes and sizes for the elements so that they will withstand the imposed loading conditions without failure.

There are many possible modes of failure, so the designer must examine the design carefully to make sure that all of the likely failure modes are covered in the analysis.

Some of the most common causes of structural failure:

1. Excessive stress - leads to fracture or permanent deformation (yielding, slip)
2. Excessive deformation – may or may not lead to fracture or permanent deformation.
3. Fatigue under repetitive or cyclic loading, initial defects or flaws grow to critical size, causing failure.
4. Instability, or buckling.
5. Creep - time-dependent deformation.
6. Others to be covered in advanced courses.
Example: Auto Front Suspension

1. Example of excessive stress:

\[ \sigma_{CD} = \frac{F_{CD}}{A_{CD}} \]

reaches level required for fracture or deformation

2. Example of excessive deformation:
3. Fatigue: Due to wheel movement and bumps in road, all loads are repetitive with time – allowable stresses are reduced.


---

**Design Problem**

A round steel rod (E = 200 GPa) 6 meters long must carry a tensile load of 7000 N. The two design constraints are:

1. The allowable stress in the steel is 125 MPa, and
2. The allowable elongation of the rod is 2.5 mm.

What is the minimum required diameter of the rod?
Stress:

\[ \sigma = \frac{P}{A} = \frac{P}{\left(\frac{\pi}{4} d^2\right)} \]

\[ d = \sqrt{\frac{4P}{\pi \sigma}} = \sqrt{\frac{4(7000)}{\pi (125 \times 10^6)}} = 0.0084 \text{ m} = 8.4 \text{ mm} \]

Elongation:

\[ \delta = \frac{PL}{AE} = \frac{PL}{\left(\frac{\pi}{4} d^2\right)E} \]

\[ d = \sqrt{\frac{4PL}{\pi \delta E}} = \sqrt{\frac{4(7000)(6)}{\pi (0.0025)(200 \times 10^9)}} \]

\[ d = 0.0103 \text{ m} = 10.3 \text{ mm} \]

Only the diameter \( d = 10.3 \text{ mm} \) will satisfy both constraints, so the elongation governs the design.
Factor of Safety (uncertainty)

\[ \text{F.S.} = \frac{\text{avg. ultimate stress}}{\text{avg. allowable stress}} = \frac{\sigma_{\text{ULT}}}{\sigma_{\text{allow}}} \]

where ultimate stress = stress at failure (material property)
allowable stress = working stress under normal use
(actual stress)

obviously, we want F.S. > 1, but if F.S. is too large, the design may not be practical.

Selection of F.S. somewhat arbitrary, but depends on such factors as variability in material properties, number of loadings, expected failure mode, uncertainty of analysis, degradation of properties during operation, importance of the member to integrity of structure.

Example: primary & secondary structure in aircraft

No substitute for experience in choosing F.S!

Typical F.S. = 2.0
If uncertain about materials, loads, etc., choose higher F.S.
If good material data base, loading well understood, use lower F.S

Dynamic load (suddenly applied) – stress is twice the stress under static load of same value.
For design against yielding, use elastic strength in Tables A-17 or A-18

\[ F \cdot S = \frac{\sigma_{\text{yield}}}{\sigma_{\text{allow}}} \]

For design against Ultimate Failure, use Ultimate Strength in Tables A-17 or A-18

\[ F \cdot S = \frac{\sigma_{\text{Ultimate}}}{\sigma_{\text{allow}}} \]
Probabilistic Design - takes into account the statistical scatter in the loads, geometrical dimensions, and material strengths in order to estimate the probability of failure.

\[ F.S. = \frac{\sigma_{ULT}}{\sigma_{allow}} \]