Combined Axial Load and Bending Moment

Example: eccentric axial loading

\[ P_d = (\text{N.A about } A) \]

\[ P = \sigma I M_y - = \sigma \]

Total stress \( \sigma = \frac{P}{A} \cdot \frac{M_y}{I} \)

\( \therefore \) Neutral axis shifts away from centroid
For the structural element shown below, determine the magnitude and location of the maximum compressive stress in section AB. The answer should be expressed in terms of $P$, $w$ and $t$.

![Diagram of structural element](image)

**Imaginary cut at section AB:**

$$
\sum M_{out} = -M_{AB} - P\left(\frac{3t}{2}\right) = 0 \quad \therefore M_{AB} = -\frac{3P}{2}$$

$$
\sigma = \frac{P_{ab}}{A_{ab}} - \frac{M_{AB}y}{I} \quad \text{bottom edge, } y = -\frac{t}{2}
$$

$$
\sigma = -\frac{P}{wt} \left(\frac{3P}{2}\right) \left(\frac{t}{2}\right) \quad \frac{12}{wt}
$$

$$
\sigma = -\frac{9P}{wt} \quad \text{on bottom edge}
$$
The post shown below is subjected to a force $F$ oriented at the angle $\theta$ as shown. The force $F$ is applied at the centroid of the post. Determine the stress $\sigma$ at point A. The answer should be expressed in terms of $F$, $\theta$, $L$, $a$, $b$ and $h$.

Free Body Diagram

Section above pt. A:

Statics Equations

\[
\begin{align*}
\sum F_x &= -F \sin \theta - P = 0 \quad \therefore P = -F \sin \theta \\
\sum M_{cut} &= M - FL \cos \theta = 0 \quad \therefore M = FL \cos \theta
\end{align*}
\]
Stress Calculation at point A:

\[
\sigma_x = \frac{P - Fy}{I} - \frac{Fl \cos \theta}{bh} - \frac{Fl \cos \theta}{bh^3 / 12}
\]

\[
\sigma_x = -\frac{Fl \cos \theta}{bh} - \frac{12 Fl \cos \theta}{bh^3}
\]

Vise Clamp Stress Analysis

If clamping force \( P = 3,000 \text{ lb} \), what is maximum stress in channel section?

\( I = 0.44 \text{ in}^4 \)
\( A = 2.12 \text{ in}^2 \)

1.72" 0.46"
1.72" - 0.46" = 1.26"
at top:

\[
\sigma = \frac{P}{A} \frac{My}{I} = \frac{3000}{2.12} - \frac{(-5130)(0.46)}{0.44} = \\
= 1415 + 5363 = 6778 \text{ psi}
\]

at bottom:

\[
\sigma = \frac{P}{A} \frac{My}{I} = \frac{3000}{2.12} - \frac{(-5130) - (1.26)}{0.44} = \\
= 1415 - 14690 = -13275 \text{ psi} = \sigma_{\text{max}}
\]
The stress at point B is:

\[
\sigma_B = \frac{P}{A} + \frac{M_c}{I} + \frac{F \cos \theta \left( \frac{d}{2} \right)}{\pi d^4}
\]

\[
\sigma_B = \frac{F \sin \theta}{\pi d^4} + \frac{F \cos \theta}{\pi d^4}
\]

consider the two extreme values of \( \theta \)

- at \( \theta = 0^\circ \), \( (\sigma_B)_0 = \frac{32 F e}{\pi d^3} \) (bending only - side loading)
- at \( \theta = 90^\circ \), \( (\sigma_B)_{90} = \frac{4 F}{\pi d^2} \) (axial load only - direct pull)
The ratio of these two extreme values is

\[
\frac{(\sigma_B)_{0^\circ}}{(\sigma_B)_{90^\circ}} = 8 \left( \frac{e}{d} \right)
\]

Since \( e > d \) for typical eye bolts, the maximum bending stress is much greater than the maximum axial stress. Thus, the eye bolt should not be used in side loading! Serious accidents have been caused by such improper use.

Concrete very weak in tension. What is location of load that would just begin to cause concrete to go into tension at base?
Stresses due to $P$

$\sigma = -\frac{P}{A}$

due to $M$

$\sigma = -\frac{M_e}{I}$

at point $A$, $\sigma = \frac{M_e}{I} = \frac{P}{A}$

when

$\sigma = \frac{(Pe)\left(\frac{h}{2}\right)}{bh^3} = \frac{P}{bh^3}$

$\therefore e = \frac{h}{6}$

when $e > \frac{h}{6}, \quad \sigma_A > 0$

when $e < \frac{h}{6}, \quad \sigma_A < 0$

Design of Dam

Design so that stress here is never tensile

water
Find stresses at point A

Section B - B

\[ \bar{C} = \frac{(4)(8)(2) + (4)(8)(8)}{(4)(8) + (4)(8)} = 5' \]

\[ \bar{C} = 5' \]

\[ I = \frac{8(4)^3}{12} + 32(3)^3 + \frac{4(8)^3}{12} + 32(3)^3 \]

\[ I = 789.3 \text{ in}^4 \]

\[ Q_A = (4)(4)(5) = 80 \text{ in}^3 \]

\[ A = 64 \text{ in}^2 \]
\[ \Sigma F_y = \frac{3}{5} (2500) - V_A = 0 \]
\[ \therefore V_A = \frac{3}{5} (2500) = 1500 \text{ lb} \]

\[ \Sigma M_{cut} = M_A - 9(12) \frac{3}{5} (2500) = 0 \]
\[ \therefore M_A = (9)(12) \frac{3}{5} (2500) = 162,000 \text{ in lb} \]

\[ \Sigma F_x = P_A + \frac{4}{5} (2500) = 0 \]
\[ \therefore P_A = -\frac{4}{5} (2500) = -2000 \text{ lb} \]

\[ (\sigma_x)_A = \frac{P_A}{A} - \frac{M v}{I} = -\frac{2000}{64} - \frac{(162,000)(3)}{789.3} \]
\[ = -647 \text{ psi} \]

\[ (r_{xy})_A = \frac{V_A Q}{It} = \frac{(+1500)(80)}{(789.3)(4)} = 38 \text{ psi} \]
Recall previous example on combined stress

we found previously that

\[
(s_{\sigma})_A = \frac{P_A}{A} \cdot \frac{M_y}{I} = -\frac{2000}{64} - \frac{(162,000)(3)}{789.3} = -647 \text{ psi}
\]

\[
(s_{\tau})_A = \frac{V_s Q}{I_t} = \frac{(+1500)(80)}{(789.3)(4)} = 38 \text{ psi}
\]

Now we want to find the principal stresses, maximum shear stress and orientation of the corresponding axes

\[
\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{-647 + 0}{2} = -324 \text{ psi}
\]

\[
R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\frac{\tau_{XY}}{2}\right)^2} = \sqrt{\left(\frac{-647}{2}\right)^2 + (38)^2} = 326 \text{ psi} = \tau_{max}
\]

\[
\sigma_{max} = \sigma_{ave} + R = -324 + 326 = 2 \text{ psi}
\]

\[
\sigma_{min} = \sigma_{ave} - R = -324 - 326 = -650 \text{ psi}
\]

\[
\theta_r = \frac{1}{2} \tan^{-1} \left( \frac{2\tau_{XY}}{\sigma_x - \sigma_y} \right) = \frac{1}{2} \tan^{-1} \left( \frac{2(38)}{647} \right) = 3.35^\circ
\]
Combined Axial, Torsional, Bending and Transverse Loads

Find stresses at A, B, C, & D
FBD: Draw in as positive T, P, V, M

Statics:
\[ \sum F_x = -P + F_2 = 0 \quad \therefore P = F_2 = 1000 \quad \text{lb} \]
\[ \sum F_y = V - F_1 = 0 \quad \therefore V = F_1 = 1000 \quad \text{lb} \]

\[ \sum M_z = T - T_1 = 0 \quad \therefore T = T_1 = 24,000 \quad \text{in} \cdot \text{lb} \]
\[ \sum M_x = M + 48F_1 = 0 \quad \therefore M = -48(1000) = -48,000 \quad \text{in} \cdot \text{lb} \]

At a point A: (on top of shaft)
\[ (\sigma_x)_A = \frac{P}{A} - \frac{M_y}{I} \]
\[ = \frac{1000}{\pi (3)^3} - \frac{-48,000}{\pi (3)^3} \]
\[ = 35 + 2.264 \]
\[ = 2299 \quad \text{psi} \]
\[ (r_A)_x = \frac{T_c}{J} = \frac{(24,000)(3)}{\pi (3)^3} = 566 \quad \text{psi} \]

(in XZ plane tangential to shaft surface)
\[
\begin{align*}
\sigma_x &= 2,300 \text{ psi} \\
\tau &= 566 \text{ psi} \\
\text{in tangential plane}
\end{align*}
\]

At a point B: (on N.A.)

\[
(\sigma_x)_{t} = \frac{P}{A} - \frac{My}{I}
= \frac{1000}{\pi (3)^2} \left( -\frac{48,000}{\pi (3)^2} \right) 0
= 35 \text{ psi}
\]

Note: 
\[
(\tau_{B})_v = \frac{VQ}{It} = 0
\]
(in vertical plane only)

since \(Q_x = 0\)

\[
A = \frac{1}{2} \pi (3)^2
= 14.14 \text{ in}^2
\]

\[
\nu = \frac{4t}{3\pi}
= \frac{4(3)}{3\pi}
= 1.273
\]

\[
(\tau_{B})_v = \frac{VQ}{It} = \frac{(1000)(1.273)(14.14)}{\pi (3)^2} = 47 \text{ psi}
\]

\[
(\tau_{B})_t = \frac{Tc}{J} = \frac{(24,000)(3)}{\pi (3)^2} = 566 \text{ psi}
\]

\[
(\tau_{B})_{\text{total}} = (\tau_{B})_v + (\tau_{B})_t = 613 \text{ psi}
\]

Both in vertical XY plane at B
At a point C:

\[ (\sigma_x)_c = \frac{P}{A} - \frac{My}{I} = \frac{1000}{\pi (3)^2} - \frac{(-48,000)(-3)}{2\pi (3)^4} \]

\[ = 35 - 2,264 = -2,229 \text{ psi} \]

\[ (\tau_c)_r = \frac{T_c}{J} = 566 \text{ psi (in XZ plane)} \]

\[ (\tau_c)_n = \frac{VQ}{It} = 0 \text{ since } Q_c = 0 \]

if \( (\tau_c)_n \) were \( \neq 0 \), it would be in XY plane and could not be added to \( (\tau_c)_r \).

At a point D: (on N.A.)

\[ (\sigma_x)_D = (\sigma_x)_a = 35 \text{ psi} \]

\[ (\tau_D)_n = (\tau_n)_r = 47 \text{ psi} \]

\[ (\tau_D)_r = 566 \text{ psi} \]

\[ (\tau_D)_{total} = 566 - 47 = 519 \text{ psi} \]
Yield Criteria for Ductile Materials

How can uniaxial tensile test data be used to predict yielding in a 2-D or 3-D state of stress?

Recall that yielding in ductile materials is due to slip and dislocation movement, which in turn is driven by shear stress. Shear stresses are generated by virtue of principal stress differences.

Example: hydrostatic stress \( \sigma_a = \sigma_b = \sigma_c = \sigma \)

Thus, several yield criteria for 2-D & 3-D states of stress are based on principal stress differences.
For uniaxial tension test

\[ \sigma_a = \sigma \]

Yielding in uniaxial test occurs when \( \sigma = \sigma_Y \), the yield stress. The corresponding max. shear stress at onset of yielding is

\[ \tau_{max} = \frac{\sigma_Y}{2} \]

The 2-D & 3-D yield criteria are all based on comparison of some measure of the principal stress differences and max. shear stresses with the corresponding measure for the uniaxial test.

**Maximum shearing stress criterion**

When the maximum shearing stress (the radius of largest Mohr’s circle) in the general 2-D or 3-D state of stress equals the max. shear stress in the uniaxial tensile test, yielding begins.

\[ \tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} \]
\[ \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} = \frac{\sigma_y}{2} \]

or

\[ \sigma_{\text{max}} - \sigma_{\text{min}} = \sigma_y \]

\[ \therefore \text{ No Yielding when } \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} < \frac{\sigma_y}{2} \]

or

\[ \sigma_{\text{max}} - \sigma_{\text{min}} < \sigma_y \]

Another way to look at it - define an equivalent axial stress for the general 2D or 3D case:

\[ \sigma_{eq} = \sigma_{\text{max}} - \sigma_{\text{min}} \]

Then yielding in the general 2D or 3D case begins when

\[ \sigma_{eq} = \sigma_Y \]

**Maximum Distortion Energy (von Mises) Criterion:**

When the root mean square (RMS) of the principal stress differences (i.e., RMS of 3 circle diameters) reaches the same value as it has when yielding begins in a tensile test, then yielding begins in 2-D or 3-D case:

General 3-D case:
General 3-D case:
\[
\text{RMS} = \sqrt{\frac{1}{3} \left[ (\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_a - \sigma_c)^2 \right]}
\]

Tensile test (\(\sigma_a\) applied)
\[
\text{RMS} = \sqrt{\frac{1}{3} \left[ (\sigma_a - 0)^2 + (\sigma_a - 0)^2 \right]} = \sqrt{\frac{2}{3} \sigma_a^2}
\]

At yield in tensile test, \(\sigma_a = \sigma_y\), and
\[
\text{RMS} = \sqrt{\frac{2}{3} \sigma_y^2}
\]

Therefore yielding in general 3-D case occurs when

\[
\sqrt{\frac{1}{3} \left[ (\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_a - \sigma_c)^2 \right]} = \sqrt{\frac{2}{3} \sigma_y^2}
\]

or
\[
(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_a - \sigma_c)^2 = 2\sigma_y^2
\]

for 2-D case, with \(\sigma_c = 0\)
\[
(\sigma_a - \sigma_b)^2 + \sigma_b^2 + \sigma_c^2 = 2\sigma_y^2
\]
\[
\sigma_a^2 - 2\sigma_a\sigma_b + \sigma_b^2 + \sigma_a^2 + \sigma_a^2 = 2\sigma_y^2
\]

or
\[
\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 = \sigma_y^2
\]

No yielding when LHS < RHS
Another way to look at it – define an equivalent axial stress (or von Mises stress) for the general 2D or 3D case:

\[
\sigma_{eq} = \sqrt{\frac{1}{2} \left[ (\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_a - \sigma_c)^2 \right]}
\]

Then yielding in the general 2D or 3D case begins when

\[
\sigma_{eq} = \sigma_Y
\]

**Yield surfaces**

Maximum Shearing Stress Criterion:

\[
\sigma_{max} - \sigma_{min} = \sigma_Y
\]

Six possibilities for plane stress (\(\sigma_c = 0\))

1) \(\sigma_a > \sigma_b > \sigma_c\) \hspace{1cm} \(\sigma_a - \sigma_c^0 = \sigma_Y\) \hspace{1cm} \(\therefore \sigma_a = \sigma_Y\)

\[
\tau_{max} = \frac{\sigma_a - \sigma_c}{2}
\]
2) \( \sigma_c > \sigma_b > \sigma_a \)

\[
\sigma_c - \sigma_a = \sigma_y \\
\therefore \sigma_a = -\sigma_y
\]

3) \( \sigma_b > \sigma_a > \sigma_c \)

\[
\sigma_b - \sigma_c = \sigma_y \\
\therefore \sigma_b = \sigma_y
\]

4) \( \sigma_c > \sigma_a > \sigma_b \)

\[
\sigma_c - \sigma_b = \sigma_y \\
\therefore \sigma_b = -\sigma_y
\]

5) \( \sigma_a > \sigma_c > \sigma_b \)

\[
\sigma_a - \sigma_b = \sigma_y
\]

6) \( \sigma_b > \sigma_c > \sigma_a \)

\[
\sigma_b - \sigma_a = \sigma_y
\]
3-D Case - hexagonal cylinder

intersects $\sigma_a$, $\sigma_b$ plane
to give hexagon above
Von Mises Ellipse

\[ \sigma_a^2 - \sigma_a \sigma_b + \sigma_b^2 = \sigma_y^2 \]

Experimental data falls closer to von Mises ellipse, but Maximum Shear criterion is more conservative

Example:

\[ \sigma = \frac{P}{A} \]
\[ \tau = \frac{T}{J} \]
\[ \tau_{\text{max}} = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \]
\[ \sigma_{\text{max}} - \sigma_{\text{min}} = \sigma_y \]

Can now establish what combinations of P & T can be used without yielding
**Brittle fracture Criteria**

Maximum Normal Stress Criterion

\[ |\sigma_a| < \sigma_U \quad \sigma_b < \sigma_U \]

where \( \sigma_U \) = Ultimate strength (assumed same in T or C)

**Mohr’s Criterion** - tensile & compressive strengths not the same
Maximum Shearing Stress Criterion applied to pressure vessel

\[ \tau_{\text{max}} = \frac{pr}{2t} = \frac{\sigma_y}{2} \]

\[ \text{or} \quad \sigma_{\text{max}} - \sigma_{\text{max}} = \sigma_y \]

\[ \therefore \quad \frac{pr}{t} - 0 = \sigma_y \]

\[ \therefore \quad p = \frac{\sigma_y t}{r} \]

is pressure that will cause yielding according to \( \tau_{\text{max}} \) criterion

A cylindrical pressure vessel with closed ends has a mean diameter of 500 mm. The tensile yield stress of the vessel material is 250 Mpa. Use the Maximum Shear Stress yield criterion to find the minimum required wall thickness, \( t \), if the internal pressure is 2 Mpa.

Stresses:

\[ \sigma_z = \frac{pr}{2t} \quad , \quad \sigma_\theta = \frac{pr}{t} \quad , \quad \sigma_r \approx 0 \]
A thin walled spherical pressure vessel of mean diameter D and wall thickness t is subjected to an internal pressure p. If the tensile yield strength of the vessel material is Y, determine the maximum allowable internal pressure in the vessel according to the
(a) Maximum Shear Stress yield criterion, and
(b) Mises yield criterion.

Answers should be expressed in terms of given parameters.
\[
\sigma_\phi = \sigma_\theta = \frac{pr}{2t}, \quad \sigma_r \approx 0 \begin{cases} 
\sigma_1 = \frac{pr}{2t} = \sigma_2 \\
\sigma_3 = 0 \end{cases}
\]

\[
\tau = \frac{pr}{2t} \sin \phi, \quad \phi, \quad \phi = \frac{pr}{4t} 
\]

(a) \[\tau_{\text{max}} = \frac{pr}{4t} = \frac{Y}{2} \quad \therefore \quad p = \frac{2Yt}{r} = \frac{4Yt}{D}\]

(b) \[\sqrt{\frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]} = Y\]

\[
\sqrt{\frac{1}{2} \left[ (0)^2 + \left( \frac{pr}{2t} \right)^2 + \left( \frac{pr}{2t} \right)^2 \right]} = Y \]

\[
\frac{pr}{2t} = Y \quad \therefore \quad p = \frac{2Yt}{r} = \frac{4Yt}{D} \]

\therefore \quad \text{both yield criteria give same result in this case}
The rectangular bar below has tensile yield strength modulus of elasticity $E$, and Poisson's ratio $\nu$. The bar is subjected to a single axial force $F$ along the $x$ direction. A strain gage is attached to the bar parallel to the "p axis" at an angle of $45^\circ$ to the horizontal as shown. What normal strain $\varepsilon_p$ should the strain gage indicate at the onset of yielding according to the Maximum Shear Stress Criterion? The answer should be expressed in terms of the appropriate combination of the given parameters.

![Diagram of the rectangular bar with a strain gage and force applied.]

The equations for the stress components and strain are shown below:

\[ \sigma_x = \frac{F}{A} = \varepsilon_x ; \quad \varepsilon_x = \frac{F}{AE} \]

Maximum Shear Stress criterion
\[ \tau_{\text{max}} = \frac{F}{2A} = \frac{\sigma_y}{2} \quad \therefore F = \sigma_y A \text{ at yield} \]

and\[ \varepsilon_x = \frac{F}{AE} \]

Hooke’s Law:
\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu \sigma_y \right] \]
\[ \varepsilon_y = \frac{1}{E} \left[ \sigma_y - \nu \frac{\sigma_y}{2} \right] \]
\[ \varepsilon_p = \frac{\sigma_y}{2E} (1 - \nu) \]
Example – Use of yield criteria

State of stress given as
and tensile yield strength = $\sigma_y = 300$ MPa

What is the factor of safety against yielding?
(a) Using Maximum Shear Stress criterion
(b) Using Von Mises Criterion

$$\text{F.S.} = \frac{\text{yield strength}}{\text{allowable stress}} = \frac{\sigma_y}{\sigma_{allow}} = \text{Factor of Safety}$$

Thus instead of using $\sigma_y$ in calculations, use $\sigma_{allow} = \frac{\sigma_y}{\text{F.S.}}$

Mohr’s circle:

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{40 + 150}{2} = 95 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{40 - 150}{2}\right)^2 + (50)^2}$$

$$R = 74.33 \text{ MPa}$$

$$\sigma_u = \sigma_{ave} + R = 95 + 74.33 = 169.33 \text{ MPa}$$

$$\sigma_b = \sigma_{ave} - R = 95 - 74.33 = 20.67 \text{ MPa}$$

$\sigma_{max} = \sigma_u = 169.33$ MPa

$\sigma_{min} = \sigma_c = 0$ (plane stress)
(a) Maximum Shear Stress Criterion:

\[ \sigma_{max} - \sigma_{min} = \frac{\sigma_y}{F \cdot S} \cdot \]
\[ 169.33 - 0 = \frac{300}{F \cdot S} \cdot \]
\[ \therefore F \cdot S. = 1.772 \]

(b) Von Mises Criterion:

\[ \sigma_a^2 - \sigma_a \sigma_b + \sigma_b^2 = \sigma_y^2 \]
\[ (169.33)^2 - (169.33)(20.67) + (20.67)^2 = \left( \frac{300}{F \cdot S.} \right)^2 \]
\[ \therefore F \cdot S. = 1.875 \]