Coupled, forced response of an axially moving strip with internal resonance

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Abstract

In this paper, the forced response of a non-linear axially moving strip with coupled transverse and longitudinal motions is studied. In particular, the response of the system is examined in the neighborhood of a 3:1 internal resonance between the first two transverse modes. The equations of motion are derived using the Hamilton’s Principle and discretized by the Galerkin’s method. First, with the longitudinal motion neglected, the forced transverse response is investigated by applying the method of multiple scales to assess the effects of speed and the internal resonance. In general, the speed is shown to affect each mode differently. The internal resonance results in the constant solutions having transition to instability of both a saddle-node type and a Hopf bifurcation. In the region where the Hopf bifurcation occurs, steady-state periodic motion does not exist. Instead the stable motion is amplitude- and phase-modulated. When the coupled system with longitudinal motion is examined with internal resonance, results reveal that the modulated motions disappear. Thus, the presence of the longitudinal motion has a stabilizing effect on the transverse modes in the Hopf bifurcation region. The second longitudinal mode is shown to drift due primarily to a direct excitation of the first transverse mode. Effects of the longitudinal motion on the transverse response are shown to be significant for speeds both away from and close to the critical speed.

Keywords: Axially moving; Coupled; Nonlinear; Internal resonance

1. Introduction

The class of axially moving materials includes many real-life systems such as band saws, belt and chain drives and magnetic tapes. Most often these systems are modeled as either a traveling tensioned beam or string. These basic models belong to the class of gyroscopic systems, which experience a divergence instability at a critical speed where the fundamental natural frequency vanishes. Linear vibration of axially moving materials has been studied extensively [1]. While a linear analysis provides natural frequencies, mode shapes and critical speeds, its validity regarding the response of the system diminishes as the vibration amplitude becomes sufficiently large or as the critical speed is approached [2]. In these cases, one must resort to a non-linear analysis.

The subject of non-linear vibrations of axially moving materials has received much attention in the literature. Mote [2] studied the non-linear axially moving string and showed the significance...
of the tension variation as velocity increases, thereby limiting linear analysis to high-tension, low-velocity problems. Bapat and Srinivasan [3] used the method of harmonic balance to determine the frequency of transverse oscillations for the non-linear traveling string. Ames et al. [4] used a finite difference technique to compute the non-linear response of a harmonically excited translating string while Kim and Tabarrok [5] solved the non-linear response of the traveling string using the method of characteristics. Using perturbation techniques, Thurman and Mote [6] studied the coupled, non-linear, partial differential equations governing the longitudinal and transverse motions of an axially moving strip. They calculated the first- and second-order approximations of the period of the system. Their results show that the presence of the transport speed increases the relative importance of the non-linear terms. Wickert [7] studied the non-linear translating beam in both the sub- and super-critical transport speed ranges. It was found that the contribution of non-linear stiffness increases with speed and grows most rapidly near critical speeds. The non-linear vibration of power transmission belts was studied by Moon and Wickert [8]. Excitation due to eccentricities at the pulleys resulted in multi-valued amplitudes and jumps for the response versus speed. Zhang and Zu [9] studied the non-linear forced vibration of viscoelastic moving belts using the method of multiple scales. The viscoelasticity was found to decrease the amplitude of vibration. Recently, Chakraborty et al. [10] studied the free and forced vibration of the non-linear traveling beam. Non-linear, non-stationary, complex normal modes were obtained and the frequency response revealed a hardening characteristic.

The non-linear vibration of a beam or string is known to have transverse and longitudinal motions, resulting in two non-linearly coupled equations governing the vibration. In many cases, the two equations are reduced to one non-linear integro-differential equation governing the transverse vibration. This is accomplished by assuming that the beam is thin thereby neglecting the longitudinal dynamics. Stated another way, it is assumed that the Green’s strain is constant along the neutral axis [11]. With this assumption, an integral equation governing the longitudinal displacement is found showing that the longitudinal motion arises entirely from finite transverse vibration. In other words, the transverse motion can affect the longitudinal motion but not the other way around; i.e., the coupling is one way. With the exception of Thurman and Mote [6], all of the above non-linear studies used the single non-linear integro-differential equation of motion of the transverse vibration. While this equation does capture the essential non-linear characteristics of the transverse vibration, it is difficult to accurately assess the complete multi-mode interaction or coupling between the longitudinal and transverse motions. To study the complete interaction between the longitudinal and transverse motions, it is necessary to use the fully coupled non-linear equations of motion. This coupling may be significant since the results of Thurman and Mote [6] indicate an increasing contribution from the longitudinal motion at high speeds.

Another issue concerning the coupling in the non-linear translating beam is the effect of an internal resonance on the response. Depending on the non-linearity, internal resonance can occur if any of the natural frequencies of the system are commensurable. For the axially moving beam, this occurs for certain combinations of tension and speed. When such a condition exists, multi-mode interaction becomes strong, and hence, significant [12,13]. The coupled non-linear response of systems with internal resonance has received considerable attention [14]. It has been shown that these types of systems can exhibit two types of unstable motion: one associated with the jump phenomenon and the other associated with an amplitude-modulated motion [15,16]. Sethna and Bajaj [17] found that the unstable response associated with the amplitude-modulated motion is due to a Hopf bifurcation. Nayfeh and Zavodney [18] confirmed the existence of the modulated motions experimentally. Using perturbation techniques, Tousi and Bajaj [19] studied the response of a system with cubic non-linearities and found that for sufficiently small damping, period-doubling bifurcations can occur which could lead to chaotic motion. Lee and Perkins [20] studied the non-linear vibration of cables with 2:1 internal resonance between the in-plane and out-of-plane modes while Nayfeh
et al. [21] formulated a general procedure for studying the forced response of structural elements with internal resonance. To the best of our knowledge, no studies exist addressing the issue of internal resonance in axially moving materials.

The purpose of this paper is to examine the coupled forced response of an axially moving strip with internal resonance. The paper is organized as follows. The non-linear partial differential equations of the axially moving strip are derived using Hamilton’s Principle and are discretized using Galerkin’s method. Using the method of multiple scales, a perturbation analysis is performed on the transverse motion. The effects of speed and a 3:1 internal resonance on the frequency response of the transverse modes are determined. Next, the frequency response of the coupled transverse and longitudinal equations is investigated numerically for both low and high speeds as well as with the internal resonance.

2. Problem formulation

Fig. 1 depicts a simply supported, axially moving strip under an applied tension $P$ with a constant transport speed $\vec{V}$. The motion of the moving strip is restricted to the $xz$ plane with transverse and longitudinal displacements denoted by $w(x,t)$ and $u(x,t)$, respectively. The displacement field is

$$\hat{u} = u(x,t) - zw_{xx}(x,t), \quad \hat{w} = w(x,t)$$

and the corresponding normal strain in the $x$ direction is

$$\varepsilon_{xx} = \frac{\hat{u}}{\hat{x}} + \frac{1}{2} \left[ \left( \frac{\hat{u}}{\hat{x}} \right)^2 + \left( \frac{\hat{w}}{\hat{x}} \right)^2 \right] + \frac{P}{EA}$$

$$\approx u_{xx} - zw_{xx} + \frac{1}{2} u_{xx}^2 + \frac{1}{2} w_{xx}^2 + \frac{P}{EA}.$$  

where $P/EA$ is the strain due to the applied tension. The shear strain $\varepsilon_{xz}$ is neglected in this analysis and all other strains are zero. The strain energy of the system is

$$U = \frac{1}{2} \int_0^L \int_A E \varepsilon_{xx}^2 \, dA \, dx = \frac{1}{2} \int_0^L \int_A E \left( u_{xx} - zw_{xx} \right)^2 \, dA \, dx$$

$$+ \frac{1}{2} u_x^2 + \frac{1}{2} w_x^2 + \frac{P}{EA} \right)^2 \, dA \, dx.$$  

(3)

Integrating (3) over the area gives

$$U = \frac{1}{2} \int_0^L \rho A \left( u_{xx} + u_{xx}^3 + u_{xx} w_{xx}^2 \right)$$

$$+ \frac{1}{2} u_{xx}^2 w_{xx}^2 + \frac{I}{A} w_{xx}^2 + \frac{4}{4} u_{xx}^4 + \frac{4}{4} w_{xx}^4$$

$$+ P \left( 2u_{xx} + u_{xx}^2 + w_{xx}^2 + \frac{P}{EA} \right) \, dx,$$  

(4)

where $I = \int_A z^2 \, dA$ and $\int_A z \, dA = 0$. The kinetic energy of the system is

$$T = \frac{1}{2} \int_0^L \int_A \rho \vec{V} \cdot \vec{V} \, dA \, dx$$

$$= \frac{1}{2} \int_0^L \left[ \{u_t + \vec{V}(1 + u_{xx}) \}^2 + \{w_{tt} + \vec{V}w_{xx} \}^2 \right] \, dx,$$  

(5)

where $\rho$ is the density and $\vec{V}$ is the absolute velocity vector of the axially moving element. The equations of motion and the associated boundary conditions are derived through the Hamilton’s Principle. Performing the variation of the integrals and integrating by parts yield the following coupled,
non-linear equations of motion:

\[
\rho A u_{s,t} + 2 \rho A V u_{x,t} + (\rho A V^2 - EA - P) u_{x,x} = E A [3 u_{x,x} u_{x,x} + \frac{3}{2} u_{x,x}^2 u_{x,x} + \frac{1}{2} u_{x,x}^2 w_x^2 + u_{x,x} w_x w_{x,x} + w_x w_{x,x}].
\]

(6)

\[
\rho A w_{s,t} + 2 \rho A V w_{x,t} + (\rho A V^2 - P) w_{x,x} + EI w_{x,x,x} = E A [\frac{3}{2} w_{x,x} u_{x,x} + \frac{3}{2} w_{x,x}^2 w_{x,x} + \frac{1}{2} w_{x,x} u_{x,x}^2 + w_{x,x} u_{x,x} + u_{x,x} w_{x,x},]
\]

(7)

with solutions satisfying the boundary conditions

\[
u(0,t) = u(L,t) = 0, \quad w(0,t) = w(L,t) = 0, \quad w_{x,x}(0,t) = w_{x,x}(L,t) = 0.
\]

(8)

In this formulation, the constancy of the Green’s strain \(\partial u / \partial x + \frac{1}{2} (\partial v / \partial x)^2\) is not assumed, and the full non-linear equations are examined. Note that these equations are similar to those in [22] derived for coupled spans. Introduce the following non-dimensional quantities

\[
u^* = \frac{u}{L}, \quad w^* = \frac{w}{L}, \quad x^* = \frac{x}{L}, \quad t^* = \frac{\rho L^2}{\rho L^2 + \omega^2}, \quad c = \frac{\sqrt{V}}{\sqrt{\rho L^2 + \omega^2}}.
\]

(10)

where \(\omega\) is a characteristic frequency. Dropping the asterisks for brevity, the non-dimensional equations of motion and boundary conditions are

\[
u_{s,t} + 2 c u_{x,t} + (c^2 - \lambda_1 - \beta) u_{x,x} = \lambda_1 (3 u_{x,x} u_{x,x} + \frac{3}{2} u_{x,x}^2 u_{x,x} + \frac{1}{2} u_{x,x}^2 w_x^2 + u_{x,x} w_x w_{x,x} + w_x w_{x,x} + u_{x,x} w_{x,x},)
\]

(11)

\[
w_{s,t} + 2 c w_{x,t} + (c^2 - \beta) w_{x,x} + \lambda_2 w_{x,x,x} = \lambda_1 (w_{x,x} u_{x,x} + \frac{3}{2} w_{x,x}^2 w_{x,x} + \frac{1}{2} w_{x,x} u_{x,x}^2 + w_{x,x} u_{x,x} + u_{x,x} w_{x,x}) + w_{x,x} u_{x,x} + u_{x,x} w_{x,x}.
\]

(12)

\[
u(0,t) = u(1,t) = 0, \quad w(0,t) = w(1,t) = 0, \quad w_{x,x}(0,t) = w_{x,x}(1,t) = 0.
\]

(13)

Applying the following separable solutions in terms of admissible functions:

\[
u(x,t) = \sum_{j=1}^{N} a_j(t) \sin(j \pi x),
\]

(15)

\[
w(x,t) = \sum_{j=1}^{M} b_j(t) \sin(j \pi x)
\]

and the Galerkin’s method to Eqs. (11), (12) leads to a set of \(N + M\) second-order ODEs. For \(N = M = 2\) (two longitudinal and two transverse modes), the equations of motion are

\[
\ddot{a}_1 - \frac{16}{5} c^2 \ddot{a}_2 + (\lambda_1 - \varphi) \pi^2 a_1 =
\]

\[
- \lambda_1 \left( \frac{6}{5} \pi^4 a_1^3 + 3 \pi^4 a_1 a_2 + 3 \pi^4 a_1 a_2^2 + \frac{6}{5} \pi^4 a_1 b_1^2 \right.
\]

\[
+ \pi^3 b_1 b_2 + 2 \pi^3 a_2 b_1 b_2 + \pi^3 a_1 b_2^2),
\]

(16a)

\[
\ddot{a}_2 + \frac{16}{5} c^2 \ddot{a}_1 + 4 (\lambda_1 - \varphi) \pi^2 a_2 =
\]

\[
- \lambda_2 \left( \frac{6}{5} \pi^4 a_2^3 + 3 \pi^4 a_2 a_1 + 6 \pi^4 a_2 b_1^2 + \frac{6}{5} \pi^4 a_2 b_1^2 \right.
\]

\[
+ \pi^3 a_1 b_2 + 2 \pi^3 a_1 b_1 b_2 + 6 \pi^4 a_2 b_1^2),
\]

(16b)

\[
\ddot{b}_1 - \frac{16}{5} c^2 b_2 + (\pi^2 \lambda_2 - \varphi) \pi^2 b_1 =
\]

\[
- \lambda_2 \left( \frac{6}{5} \pi^4 b_1^3 + 3 \pi^4 b_1 a_2 + 3 \pi^4 b_1 b_2^2 + \frac{6}{5} \pi^4 b_1 b_2^2 \right.
\]

\[
+ \pi^3 b_1 a_2 + 2 \pi^3 a_1 a_2 b_2 + \pi^3 a_1 b_2^2),
\]

(16c)

\[
\ddot{b}_2 + \frac{16}{5} c^2 b_1 + 4 (\pi^2 \lambda_2 - \varphi) \pi^2 b_2 =
\]

\[
- \lambda_2 \left( 6 \pi^4 b_2^3 + \pi^3 a_1 b_1 + 3 \pi^4 b_1 b_2^2 + \pi^4 a_2 b_2^2 \right.
\]

\[
+ 6 \pi^4 a_2 b_2 + 2 \pi^5 a_1 a_2 b_1),
\]

(16d)

where \(\varphi = \pi^2 - \beta\).

3. Linear eigenvalue problem

The natural frequencies of the system are determined by linearizing the above equations of motion about the straight equilibrium position. Setting the right-hand sides of (16a)–(16d) to zero, the equations of motion can be written as

\[
M \ddot{q} + G \dot{q} + K q = 0,
\]

(17)
where

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
0 & -\frac{16}{3}c & 0 & 0 \\
\frac{16}{3}c & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{16}{3}c \\
0 & 0 & \frac{16}{3}c & 0 \\
\end{bmatrix}, \quad q = \begin{bmatrix}
a_1 \\
a_2 \\
b_1 \\
b_2 \\
\end{bmatrix}
\]

\[
K = \begin{bmatrix}
(\lambda_1 - \varphi)\pi^2 & 0 & 0 & 0 \\
0 & 4(\lambda_1 - \varphi)\pi^2 & 0 & 0 \\
0 & 0 & (\pi^2\lambda_2 - \varphi)\pi^2 & 0 \\
0 & 0 & 0 & 4(4\pi^2\lambda_2 - \varphi)\pi^2 \\
\end{bmatrix}
\]

for (17) are

\[
\omega^4 - (k_{ii} + k_{jj} + g_{ij}^2)\omega^2 + k_{ii}k_{jj} = 0,
\]

where \(i = 3, j = 4\) for \(\omega_1^2, \omega_2^2\), \(i = 1, j = 2\) for \(\omega_3^2, \omega_4^2\) and \(g_{ij}, k_{ij}\) are the corresponding elements of the matrices \(G\) and \(K\), respectively. In this paper, all speeds considered are below the critical speed of the first transverse mode. Fig. 2 shows the relationship of \(\omega_2/\omega_1\) versus speed for different values of \(\beta\) (tension). It is seen that the ratio can be commensurable, possibly resulting in an internal resonance between the first two transverse modes. In general, for a larger initial tension, internal resonance occurs at a higher transport speed. In subsequent analyses, suitable parameter values can be chosen such that \(\omega_2 \approx 3\omega_1\) and the corresponding natural frequencies are used in the perturbation analysis in the next section. It should be noted that the parameters used in this study are representative of a belt drive system.

4. Forced transverse response

Since the internal resonance occurs between the transverse modes, initially we will neglect the
longitudinal motion and study only the transverse response. This will allow us to study the fundamental phenomena and characteristics of the response associated with the internal resonance. Later, the longitudinal motions are included and their effects on the transverse response are studied. The non-linearities are assumed to be weak.

4.1. Perturbation analysis

Introducing some modal damping to the system and an external excitation to the first mode, the equations of motion (16c), (16d) become

\[
\begin{align*}
\dot{b}_1 + g_{34} b_2 + k_{33} b_1 &= -d_1 \dot{b}_1 - \frac{3}{2} i \lambda_i \pi^4 b_1^2 + F_1 \cos \gamma_1 t, \\
\dot{b}_2 + g_{43} \dot{b}_1 + k_{44} b_2 &= -d_2 \dot{b}_2 - 6 \lambda_i \pi^4 b_2^2 + \frac{3}{2} i \lambda_i \pi^4 b_1^2.
\end{align*}
\] (20a)
\[
\begin{align*}
\dot{b}_1 &= i b_{11} (T_0, T_2) + i^3 b_{13} (T_0, T_2) + \cdots, \\
\dot{b}_2 &= i b_{21} (T_0, T_2) + i^3 b_{23} (T_0, T_2) + \cdots.
\end{align*}
\] (21a)

Here we are interested in the response and stability of the modes when \( \gamma_1 \approx \omega_1 \) and \( \omega_2 \approx 3 \omega_1 \). The method of multiple scales is applied to determine a uniform expansion of the solutions of (20a), (20b) near primary resonance of the first transverse mode. Since all non-linearities are of order 3, solutions for \( b_1, b_2 \) have the form [12,23]

\[
\begin{align*}
b_1 &= \varepsilon b_{11} (T_0, T_2) + \varepsilon^3 b_{13} (T_0, T_2) + \cdots, \\
b_2 &= \varepsilon b_{21} (T_0, T_2) + \varepsilon^3 b_{23} (T_0, T_2) + \cdots.
\end{align*}
\] (21b)

where \( \varepsilon \) is a small parameter, \( T = t \) is a fast time scale characterizing motions with frequencies \( \omega_1, \omega_2 \) and \( T_2 = \varepsilon^2 t \) is a slow time scale characterizing the response due to the non-linearities, damping, and external excitation. The time derivatives become

\[
\frac{d}{dt} = \frac{\partial}{\partial T_0} \left( \frac{\partial T_0}{\partial t} \right) + \frac{\partial}{\partial T_2} \left( \frac{\partial T_2}{\partial t} \right) = D_0 + \varepsilon^2 D_2,
\]

\[
\frac{d^2}{dt^2} = D_0^2 + 2 \varepsilon D_0 D_2 + \varepsilon^4 D_2^2.
\] (22)

To have the damping and the external excitation appear in the same perturbation equations as the non-linearities, it is necessary to order the damping coefficients and the excitation amplitude as

\[
\begin{align*}
d_1 &= \varepsilon^2 \tilde{d}_1, & d_2 &= \varepsilon^2 \tilde{d}_2, & F_1 &= \varepsilon^2 \tilde{F}_1.
\end{align*}
\] (23)

Substituting (22) and (23) into (20a), (20b) and collecting terms of like powers in \( \varepsilon \) give, for \( \varepsilon^0 \)

\[
\begin{align*}
D_0 \dot{b}_{11} + g_{34} D_0 b_{21} + k_{33} b_{11} &= 0, \\
D_0 b_{21} + g_{43} D_0 b_{11} + k_{44} b_{21} &= 0
\end{align*}
\] (24a)

and for \( \varepsilon^1 \)

\[
\begin{align*}
D_0 \dot{b}_{13} + g_{34} D_0 b_{23} + k_{33} b_{13} &= \\
&- 2D_0 D_2 b_{11} - g_{34} D_3 D_2 b_{21} - 3 \lambda_i \pi^4 b_{11} - \tilde{F}_1 \cos \gamma_1 T_0, \\
D_0 \dot{b}_{23} + g_{43} D_0 b_{13} + k_{44} b_{23} &= \\
&- 2D_0 D_2 b_{21} - g_{43} D_3 D_2 b_{11} - \tilde{F}_2 \cos \gamma_1 T_0
\end{align*}
\] (25a)

The solutions to the homogeneous coupled equations of (24a), (24b) are

\[
\begin{align*}
b_{11} &= A_1(T_2)e^{i \omega_1 T_0} + A_2(T_2)e^{i \omega_2 T_0} + \text{cc}, \\
b_{21} &= h_1 A_1(T_2)e^{i \omega_1 T_0} + h_2 A_2(T_2)e^{i \omega_2 T_0} + \text{cc},
\end{align*}
\] (26a)

where \( \text{cc} \) stands for the complex conjugate and

\[
h_j = \frac{-\omega_j^2 + k_{33}}{g_{43} \omega_j}, \quad j = 1, 2.
\] (27)

Substituting (26a), (26b) into (25a), (25b) results in two coupled equations governing \( b_{13} \) and \( b_{23} \) having the form

\[
\begin{align*}
D_0 \dot{b}_{13} + g_{34} D_0 b_{23} + k_{33} b_{13} &= Q_1, \\
D_0 \dot{b}_{23} + g_{43} D_0 b_{13} + k_{44} b_{23} &= Q_2.
\end{align*}
\] (28a)

where \( Q_1 \) and \( Q_2 \) are the inhomogeneous terms which are functions of the amplitudes \( A_1, A_2 \) and \( \text{cc} \). The solvability conditions, which are determined by eliminating the secular and small divisor terms from (28a), (28b) give the solutions for \( A_1 \) and \( A_2 \). Since the homogeneous solutions of (28a), (28b) are proportional to both \( e^{i \omega_1 T_0} \) and \( e^{i \omega_2 T_0} \), inhomogeneous terms proportional to both \( e^{i \omega_1 T_0} \) and \( e^{i \omega_2 T_0} \) will produce secular terms in \( b_{13} \) and \( b_{23} \). To eliminate these secular terms,
a particular solution is sought of the form [23]

\[
\begin{align*}
 b_{13} &= P_1 (T_2) e^{i \omega_1 T_0} + P_2 (T_2) e^{i \omega_2 T_0}, \quad (29a) \\
 b_{23} &= S_1 (T_2) e^{i \omega_1 T_0} + S_2 (T_2) e^{i \omega_2 T_0}. \quad (29b)
\end{align*}
\]

Substituting (29a), (29b) into (28a), (28b) and retaining only the secular terms on the right-hand side give

\[
\begin{align*}
&\left( - \omega_1^2 P_1 + g_{34} i \omega_1 S_1 + k_{33} P_1 \right) e^{i \omega_1 T_0} \\
&\quad + \left( - \omega_2^2 P_2 + g_{34} i \omega_2 S_2 + k_{33} P_2 \right) e^{i \omega_2 T_0} = ST_1, \\
&\left( - \omega_1^2 S_1 + g_{43} i \omega_1 P_1 + k_{44} S_1 \right) e^{i \omega_1 T_0} \\
&\quad + \left( - \omega_2^2 S_2 + g_{43} i \omega_2 P_2 + k_{44} S_2 \right) e^{i \omega_2 T_0} = ST_2,
\end{align*}
\]

where \( ST_1 \) and \( ST_2 \) are the secular terms associated with \( Q_1 \) and \( Q_2 \), respectively. Equating the coefficients of \( e^{i \omega_1 T_0} \) and \( e^{i \omega_2 T_0} \) terms on both sides of (30a), (30b) yields four algebraic equations, two governing \( P_1 \) and \( S_1 \), and two governing \( P_2 \) and \( S_2 \), which are

\[
\begin{align*}
&\left[ - \omega_1^2 + k_{33} \right] - g_{34} i \omega_1 \left[ P_1 \right] \\
&\left[ g_{43} i \omega_1 \right] - \omega_1^2 + k_{44} \left[ S_1 \right] = \left( \hat{s}_{11} \right), \\
&\left[ - \omega_2^2 + k_{33} \right] - g_{34} i \omega_2 \left[ P_2 \right] \\
&\left[ g_{43} i \omega_2 \right] - \omega_2^2 + k_{44} \left[ S_2 \right] = \left( \hat{s}_{12} \right),
\end{align*}
\]

where \( \hat{s}_{ij} \) are the secular terms corresponding to \( e^{i \omega_j T_0} \) from \( ST_j \). From (19), it is seen that the homogeneous parts of (31) have a nontrivial solution (the coefficient matrices are singular). The solvability conditions are therefore

\[
\begin{align*}
&\left| - \omega_1^2 + k_{33} \right| \hat{s}_{11} = 0, \\
&\left| g_{43} i \omega_1 \right| \hat{s}_{12} = 0, \\
&\left| - \omega_2^2 + k_{33} \right| \hat{s}_{21} = 0, \\
&\left| g_{43} i \omega_2 \right| \hat{s}_{22} = 0,
\end{align*}
\]

yielding two complex ordinary differential equations in terms of \( A_1, A_2 \) and cc. These equations are solved by assuming a solution for \( A_1 \) and \( A_2 \) in polar form, i.e.,

\[
A_1 = \frac{1}{2} q_1 e^{i \theta_1}, \quad A_2 = \frac{1}{2} q_2 e^{i \theta_2},
\]

where \( q_1, q_2, \theta_1, \theta_2 \) are real-valued functions of time. Substituting (33) into (32) and separating the resulting equations into real and imaginary parts yield four real ordinary differential equations for \( q_1, q_2, \theta_1, \theta_2 \).

To study the 3:1 internal resonance due to the presence of cubic nonlinearities and to describe the nearness of \( \gamma_1 \) to \( \omega_1 \) (primary resonance), detuning parameters \( \sigma_1, \sigma_2 \) are introduced, i.e.,

\[
\begin{align*}
\omega_2 &= 3 \omega_1 + \varepsilon^2 \sigma_1, \quad (34) \\
\gamma_1 &= \omega_1 + \varepsilon^2 \sigma_2. \quad (35)
\end{align*}
\]

From (34), it is seen that in addition to terms proportional to \( e^{i \omega_1 T_0} \) and \( e^{i \omega_2 T_0} \) producing secular terms, terms proportional to \( e^{i (\omega_1 - 2 \omega_1) T_0} \) and \( e^{i 3 \omega_1 T_0} \) also produce secular terms in (28a), (28b). Taking these secular terms from (28a), (28b) and applying (26)–(30) gives the following equations:

\[
\begin{align*}
&\frac{1}{2} \dot{F}_1 g_{43} \omega_1 \sin \eta_1 + J_1 q_1 + J_2 q_2^{2} q_2 \sin \eta_2 \\
&\quad + J_3 q_2^{4} = 0, \quad (36a) \\
&\frac{1}{2} \dot{F}_1 g_{43} \omega_1 \cos \eta_1 + J_4 q_1^{3} + J_5 q_2^{6} q_2 \cos \eta_2 \\
&\quad + J_6 q_2^{7} q_2^{3} + J_7 q_2^{9} q_2^{3} = 0, \quad (36b) \\
&K_1 q_2 + K_2 q_2^{3} \sin \eta_2 + K_3 q_2^{2} = 0, \quad (36c) \\
&K_4 q_2^{3} \cos \eta_2 + K_5 q_2^{5} q_2 + K_6 q_2^{5} + K_7 q_2^{7} q_2^{3} = 0, \quad (36d)
\end{align*}
\]

where \( J_1 - J_7, K_1 - K_7 \) are real constants and

\[
\begin{align*}
\eta_1 &= \sigma_2 T_2 - \theta_1, \quad \eta_2 = \sigma_1 T_2 - 3 \theta_1 + \theta_2. \quad (37)
\end{align*}
\]

In (36a)–(36d), constant solutions (fixed points) correspond to steady-state periodic solutions of (20a), (20b), while periodic solutions correspond to modulated motions of the transverse modes. For all of the perturbation results that follow, \( \dot{F}_1 = 75, \) \( d_1 = d_2 = 20, \) \( \varepsilon = 0.04. \)

4.2. Steady-state periodic solutions

To determine the constant solutions of (36a)–(36d), the derivatives are set to zero resulting in four algebraic equations. These four algebraic equations can be combined into two equations which are then solved numerically for \( q_1 \) and \( q_2 \). Once \( q_1 \) and \( q_2 \) are found, the modal amplitudes \( b_1 \) and \( b_2 \) can be determined. Rewrite the system of Eqs. (36a)–(36d) as

\[
\begin{align*}
\dot{y}' &= \Lambda (y), \quad y = \{ q_1, q_2, \theta_1, \theta_2 \}^T.
\end{align*}
\]
The stability of a constant solution $y_0$ is determined by the eigenvalues of the Jacobian matrix $J = \partial A / \partial y_{0a}$. For $y_0$ to be stable, all eigenvalues of $J$ must have negative real parts. According to [17,19], there are two kinds of unstable constant solutions. The first kind corresponds to one real eigenvalue passing through the origin into the right half complex plane. This transition to instability is exhibited in frequency response curves as a “jump” phenomenon of saddle-node instability. The second kind has a pair of unstable complex eigenvalues. This corresponds to a Hopf bifurcation and solutions of the perturbation equations (36a)-(36d) are expected to have a limit-cycle with the response of (20a), (20b) executing modulated oscillations.

The response of the system away from internal resonance is considered first. Fig. 3 shows typical response curves for the steady-state solutions of the first and second modes as a function of the excitation detuning parameter. Results from both the perturbation analysis and the numerical integration of (20a), (20b) are presented. The numerical solutions were obtained by integrating (20a), (20b) using Mathematica until steady-state conditions were reached, which was determined by observing both the time histories and the phase diagrams. Once a steady-state solution was found, it was used as the initial condition to determine the steady-state solution at the next frequency. From Fig. 3, it is seen that there is very good agreement between the two results. The unstable constant solutions are of the first kind as evidenced by the jump phenomenon. In general, the system behaves as a hard spring. The general characteristics of (20a), (20b) due to the non-linearities in frequency ranges away from internal resonance are of the Duffing’s type which is well known and as such, will not be addressed here. However, since the equations are linearly coupled through the translation speed, it is necessary to determine what effects the speed, and hence the linear coupling, has on the response of the system.

To accomplish this, the maximum amplitudes of the first and second modes versus speed are shown in Fig. 4 for a fixed detuning parameter of $\sigma_2 = 500$. It should be noted that the natural frequencies of the system are different for each $c$. The plots illustrate that as the speed increases, the amplitude of the first mode decreases while the amplitude of the second mode increases. In other words, increasing the speed has a hardening effect on the first mode and a softening effect on the second mode. Thus the speed has a different effect on each mode. While each mode behaves differently as $c$ changes, plots of the response of the system (not shown) show a hardening affect due to the speed, which agrees with [10]. It is also seen that the amount of change in the amplitude of each mode is quite different. For a given change in speed, the resulting percentage increase in the response of the second mode is significantly higher than the corresponding decrease of the first mode. Therefore the speed, through the linear coupling, can have different

![Fig. 3. Frequency response of the transverse modes for $c = 0.167$, $\beta = 2.942$, $\dot{z}_1 = 1124$, $\dot{z}_2 = 1.666 \times 10^{-4}$; (—) perturbation (stable); (---) perturbation (unstable — first kind); (==) numerical simulation: (a) first mode; (b) second mode.](image-url)
quantitative and qualitative effects on the frequency response of each mode.

The response of the system near internal resonance is now examined. In Figs. 5 and 6, both the perturbation and numerical solutions are plotted and good agreement between these results is observed. The figures show how the frequency response curves change as the internal resonance is approached. When \( \omega_2 = 2.9\omega_1 \) (Fig. 5), the response of both modes resembles those of the case away from the internal resonance with the exception of the emergence of a small unstable region at \( \sigma_2 \approx -50 \). As \( \sigma_1 \to 0 \), this unstable manifold becomes larger and the response of each mode changes drastically with the overall shape and structure of the curves becoming much more complicated, see Fig. 6 for \( \omega_2 = 3.0\omega_1 \) case. Looking at the mode \( b_1 \) (Fig. 6a), it is seen that there are four distinct stable branches (A–D) and three unstable branches. Comparing with Fig. 3, there is a second multivalued region beginning at \( \sigma_2 \approx 66 \). From the eigenvalues of \( J \), the unstable branches between A and B, C and D are of the first kind (saddle-node instability) characterized by the jump behavior. However, the branch E \( (84 \leq \sigma_2 \leq 115) \) is unstable with a transition of the second kind. In other words, as the detuning is decreased from some large enough value, the branch C becomes unstable at \( \sigma_2 \approx 115 \) by the Hopf bifurcation. This instability persists until \( \sigma_2 \approx 84 \) when the branch E stabilizes again by the same process. For the mode \( b_2 \) (Fig. 5b), the same kinds of jump and bifurcation phenomena occur at the same detuning values.
In branch E, extensive numerical simulations are needed to reveal the characteristics of the response. In the next section, some numerical results are presented to highlight the main features of this region. It is also noted that, although the first mode is directly excited, the amplitudes of $b_1$ and $b_2$ are comparable near internal resonance, in contrast to the case away from internal resonance (see Fig. 3). Hence, for this system, an excitation of the fundamental mode may produce significant responses in higher modes when they are coupled through an internal resonance. This is also observed in Figs. 1 and 2 of [19].

4.3. Solutions in branch E

In this region, constant solutions of (36a)-(36d) cannot be sustained. Eqs. (20a), (20b) are thus integrated numerically, using the perturbation results as initial conditions, to determine the amplitudes of stable non-constant solutions (limit cycles). Fig. 7 plots the frequency response with constant solutions predicted by perturbation results shown as dashed curves for reference. The perturbation results predict unstable motions for $84 \leq \sigma_2 \leq 115$ while numerical integration gives modulated responses for $82 \leq \sigma_2 \leq 112$, which is in good agreement.
In Fig. 7, it is seen that the amplitudes start to increase at \( \sigma_2 \approx 82 \) and continue to increase until \( \sigma_2 \approx 107 \) where the amplitudes jump down and continue to decrease. This jump indicates a transition to instability. While the interest here is in the stable solutions, methods [25] are available to transform the differential equations (36a)–(36d) into algebraic equations in order to obtain the unstable non-constant solutions. The stability of the numerical solutions can be confirmed by obtaining the Floquet multipliers from the monodromy matrix of (20a), (20b). For the given parameters, the plots show that the stable limit cycles span almost the entire region of branch E. In fact, for sufficiently large damping, all non-constant solutions are stable. However, for sufficiently small damping, some non-constant solutions could become unstable through flip bifurcations, possibly leading to chaotic motion. Further discussion on the effect of damping on the stability of the response can be found in [19].

To show what type of response occurs in this region, a typical time history of the second mode is presented in Fig. 8a (for \( \sigma_2 = 95 \)) where the motion is shown to be amplitude-modulated. Numerical integration of the perturbation equations (36a)–(36d) shows that the phase is also modulated. To gain a better insight into the structure of the response, the corresponding power spectral density is shown in Fig. 8b. Instead of having only two distinct peaks, one corresponding to \( \gamma_1 \) and the other \( 3\gamma_1 \), each dominant peak has side frequencies. These sideband frequencies, which are a direct result of the non-linear interactions, indicate that the motion is quasi-periodic, and because the sidebands are not symmetric about the main peaks, the response is both amplitude- and phase-modulated [24]. Fig. 8b also shows that \( 3\gamma_1 \) can be the dominant frequency in this branch. This is in contrast to the other stable branches (such as A) where \( \gamma_1 \) is the dominate frequency for both \( b_1 \) and \( b_2 \).

The results here show that internal resonance can significantly affect the response of the system. Other studies [12,21] of beam systems have also demonstrated complex behavior in the presence of an internal resonance as well as differences in the response of the system away from and near an internal resonance.

5. Coupled transverse/longitudinal motion

In this section the coupled transverse and longitudinal equations (16a)–(16d) are studied. With damping added to only the transverse modes, these equations are numerically integrated to give steady-state response. Thus, only stable steady-state solutions are presented. In general, the frequency response curves for the transverse modes away from internal resonance (not shown) display the same trends as those of Figs. 3 and 4. Fig. 9 plots the frequency response curves of \( b_1 \) with and without the contribution of the longitudinal motion. Two sets of plots are shown, one at 10% of the critical speed, the other at 90% of the critical speed. For the lower branch, both plots show that the contribution of the longitudinal motion to the transverse response is quite small. The upper branch in both plots, however, clearly shows that the longitudinal motion can have a significant effect on the transverse motion. Inclusion of the longitudinal motion is shown to increase the transverse response by as much as 15%. For the free response of an axially moving strip, Mote and Thurman [6] indicated that the contribution of the longitudinal motion increases as speed increases. However, the results here for the forced response show that the contribution of the longitudinal motion is not significantly affected by the speed. Results similar to those in Fig. 9 are also found for the second transverse mode.

With results shown in Fig. 9 at 90% of the critical speed (based on a straight equilibrium), it should be noted that at such speeds, imperfections (due to non-trivial equilibrium solutions or initial curvature) could have important effects on the coupled forced response. Sources of initial curvature include bending moments generated from supporting pulleys and wheels as in the case of band saws, sag due to gravity, material imperfections and guide misalignments [26]. Wang [27] investigated the dynamic instability of a high-speed translating band with end curvatures due to applied bending moments at the supporting wheels. Without the applied moments, it was shown that there exists a pitchfork bifurcation and the trivial equilibrium becomes unstable at the classical critical speed. However, with end curvatures included,
the translation speed required to first generate multiple equilibria increases and the trivial (straight) equilibrium can no longer be sustained. Also, there is a gradual change in the equilibrium configurations even at sub-critical speeds. Hwang and Perkins [26,28] modeled a non-linear axially moving beam with arbitrary initial curvature and studied the associated equilibrium solutions and their stability.

As noted in [26,27], the occurrence of a deviation from the trivial equilibrium configuration generally requires a physical mechanism modeled in the problem, such as bending moments at the supporting pulleys. To further explain this point for our
problem, we first note the following. Based on Fig. 4 of [27], the nontrivial equilibria (without end curvature) may be approximated by

\[ u_e \approx u_0 \sin(2\pi x), \quad w_e \approx w_0 \sin(\pi x). \]  

(39)

Since \( u_0 \ll w_0 \), by assuming that \( u_e \approx 0 \) and employing a weak form solution approach to the static equations of (11) and (12), an approximate solution for \( w_0 \) is obtained

\[ w_0 \approx \sqrt{\frac{8}{3\lambda_1 \pi^2} (c^2 - \beta - \lambda_2 \pi^2)}. \]  

(40)

The above states that a non-trivial equilibrium will exist only if \( c > c_{cl} = \sqrt{\beta + \lambda_2 \pi^2} \) (the classical critical speed), i.e., the supercritical speed region. This conclusion is consistent with the finding of [27]. Thus, based on our model (no physical source of imperfections), only the trivial (straight) equilibrium position exists below the critical speed. Hence, to study the effects of imperfections on the coupled response, different equations of motion must be formulated which account for the physical imperfections. Since the focus of this paper is on internal resonance (which can occur even at speeds well below the classical critical speed, see Fig. 2), it is the subject of a future work to carefully examine the effects of imperfections on the coupled response.

The response of the system with the internal resonance is now examined. Fig. 10 shows the response curves of \( b_1 \) for the coupled transverse/longitudinal system with \( \omega_2 = 3.0 \omega_1 \). Comparing Figs. 10 and 6a, it is seen that the presence of the longitudinal motion does not change the basic jump characteristics of the transverse modes. However, the graphs also show that the unstable region \( E \) in Fig. 6a is now stable in Fig. 10. That is, with the longitudinal motion included, the response in this region is no longer modulated; a stable steady-state motion exists. Thus, the longitudinal motion appears to have a stabilizing effect on the system in the neighborhood of the internal resonance. It should be noted that in a real physical system, transverse and longitudinal motion (however small) always exists simultaneously. Thus, the results here suggest that the unstable region \( E \) is practically unrealizable. Also, from Fig. 6, the frequency range of branch \( E \) is rather small, and hence very small frequency steps are needed in experiments in order to observe the modulated motions [18].

To investigate why region \( E \) becomes stable, the frequency response of the first longitudinal mode near internal resonance is shown in Fig. 11. Comparing Figs. 10 and 11, it is seen that both have three stable branches. However, while the transverse mode jumps down (up) from the upper branch to the middle branch, the longitudinal mode jumps up (down). This observation may explain why the unstable region \( E \) in Fig. 6a becomes stable when the longitudinal motion is introduced. With the longitudinal motion included, the equations of motion contain both quadratic and cubic non-linearities, allowing energy to be possibly transferred...
from lower to higher modes and vice versa [12]. Thus, it appears that the jump down (up) of the transverse modes results in a transfer of energy to the longitudinal modes causing them to jump up (down). It is believed that this energy transfer to the longitudinal modes is part of the reason for the transverse modes being able to achieve steady-state conditions.

For the second longitudinal mode, a typical time history with the internal resonance is shown in
Fig. 12. Time history response of the second longitudinal mode for $\gamma_1/\omega_1 = 1.0, \omega_2 = 3.0\omega_1$. Other parameters are the same as those in Fig. 6.

Fig. 12. It is noted the response does not vibrate about the static equilibrium position. This type of motion, usually referred to as drift or steady streaming, is due to even-powered non-linear terms in the equations of motion [12]. It is also observed that the second longitudinal mode is the only mode that experiences this phenomenon, which is believed to be due primarily to the $\frac{1}{2}\pi^2 b_1^2$ term in (16b). Numerical results show that removing this term eliminates the drift. This term in (16b) comes from the $\frac{1}{2}u_\alpha^2$ in the strain in (2). This strain term is usually neglected because it is assumed to be quite small. However, while this strain term may be small, the results here show that the effect of this term on the forced response can be significant when the first transverse mode ($b_1$) is directly excited.

6. Summary and conclusions

A theoretical model for the coupled, forced response of a nonlinear, axially moving strip with a 3:1 internal resonance between the first two transverse modes was developed and studied. Results for the coupled transverse modes show that both a saddle-node bifurcation and a Hopf bifurcation exist for this system. In the region of the Hopf bifurcation, the motion is amplitude- and phase-modulated with the power spectral density having sideband frequencies. Results also reveal that the excitation of the fundamental mode can produce significant responses in higher modes when the internal resonance is present. The speed is shown to have a quantitative as well as qualitative effect on each mode. With the longitudinal motion included, results show that the modulated motions no longer exist for the coupled transverse/longitudinal system. This is due in part to energy being transferred from the transverse modes to the longitudinal modes. The second longitudinal mode is shown to exhibit drifting as a result of the forced excitation of the first transverse mode. The effect of the longitudinal motion on the forced response of the transverse mode can be significant for all speeds in the sub-critical speed range.

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