Analysis of Discontinuous Fiber Reinforced Lamina

- Fiber length effects not previously considered. 3 types to be considered:
  1. Aligned discontinuous fibers
  2. Off-axis aligned discontinuous fibers
  3. Randomly oriented discontinuous fibers

Analysis begins with simplest case of aligned discontinuous fibers.

Types of Discontinuous Fiber Reinforcement

(a) Aligned discontinuous fibers
(b) Off-axis aligned discontinuous fibers
(c) Randomly oriented discontinuous fibers

Representative Volume Elements for Aligned Discontinuous Fiber Composite

(a) Matrix material included at ends of fiber
(b) Matrix material not included at ends of fiber

Three Dimensional and Two Dimensional Random Orientations of Fibers

(a) Fiber length is less than thickness of part. So fibers are randomly oriented in three dimensions
(b) Fiber length is greater than thickness of part. So fiber are randomly oriented in only two dimensions.

Schematic Representation of Matrix Shear Deformation in a Short Fiber Composite

Before Deformation

After Deformation

Reinforcement of Matrix by Fiber

Fiber stiffness >> Matrix Stiffness
Consider free body diagram in fig. 6.4:
\[ \sum F_x = \left( \sigma_f + d\sigma_f \right) \frac{md^2}{4} - \sigma_f \frac{md^2}{4} - \tau mx dx = 0 \] (6.1)

Simplifying and rearranging:
\[ \frac{d\sigma_f}{dx} = -\frac{4\tau}{d} \] (6.2)

Separating variables and integrating
\[ \int_{\sigma_o}^{\sigma_f} d\sigma_f = \frac{4\tau}{d} \int_0^x dx \] (6.3)

Assume that stress transferred across the end of fiber \((\sigma_o = \sigma_f @ x = 0)\) is negligible:
\[ \sigma'_o = \frac{4\tau}{d} \int_0^x dx \] (6.4)

Thus, in order to determine \(\sigma_f\) we need to know \(\tau\) as a function of \(x\).

Two approaches to estimating \(\tau(x)\):
1. Kelly-Tyson Model – assume matrix is rigid – plastic (Fig. 6.5 (a))
2. Cox Model – assume matrix is linear elastic (Fig. 6.5(b))

For illustrative purposes, consider Kelly-Tyson Model.
Matrix shear stress \(\tau(x) = \frac{r_y}{G_m} = \text{yield stress constant}\)
\[ \therefore \sigma_f = \frac{4\tau}{d} \int_0^x dx = \frac{4r_y x}{d} \] (6.5)

And \(\sigma_f\) is linear function of \(x\).

For continuous fiber composite
\[ \epsilon_{f1} = E_{c1} \]
\[ \frac{\sigma_{f1}}{E_{f1}} = \frac{\sigma_{c1}}{E_{c1}} \]
\[ \therefore \sigma_{f1} = \frac{E_{f1}}{E_{c1}} \sigma_{c1} \]

As \(L \rightarrow L_i\) (Ineffective length)
\[ \sigma_{fmax} \rightarrow \frac{E_{f1}}{E_{c1}} \sigma_{c1} \]
Effect of fiber length on stress distributing along fiber according to Kelly – Tyson model

Variation of interfacial shear stress, $\tau$, and fiber normal stress, $\sigma_f$, with distance along the fiber according to Cox model.

Substituting $L = L_i$; $\sigma_{f,\text{max}} = \frac{E_f}{E_i} \sigma_i$.

In Eq. (6.6)

$$L_c = \frac{d \sigma_i}{\tau_y}$$  

$L_i$, referred to as “ineffective length” because $\sigma_i < \sigma_{f,\text{max}}$ over this portion of fiber. $L_c$ also referred to as “load transfer length” because interfacial shear load is transferred over this length.

Another limiting value of $\sigma_{f,\text{max}}$ is the fiber tensile strength, $s_f^{(+)}$. From Eq. (6.6), the fiber length for this maximum stress is,

$$L_c = \frac{ds_f^{(+)}}{2\tau_y}$$  

$L_c$ = critical length

Variation of interfacial shear stress, $\tau$, and fiber normal stress, $\sigma_f$, with distance along the fiber according to Kelly – Tyson model.

Predicted shear stress distributions along fiber from finite element analysis and Cox model. From Hwang [6.10].

if $L < L_c$, matrix failure occurs first
if $L > L_c$, fiber failure occurs first

Interfacial shear strength corresponding to critical length

$$\tau_y = \frac{ds_f^{(+)}}{2L_c}$$  

Thus, by measuring $L_c$, $d$ and $s_f^{(+)}$, we can determine interfacial shear strength. Drzal, et al (MSU).

Note: Measured values of $L_c$ statistically distributed

Single fiber specimen
Longitudinal Modulus of Aligned Discontinuous fiber Composite

Cox Model:
Recall Eq. (6.2):
\[ \frac{d\sigma_f}{dx} = \frac{A\tau}{d} \]
\[ . \] Rate of change of axial load in fiber is linear function of \( \tau \).
Cox assumed \( \tau \propto \frac{dP}{dx} \) and that
\[ \frac{dP}{dx} = H(u - v) \] (6.10)
where \( u \) = axial displacement at a point in fiber
\( v \) = axial displacement the matrix would have at same point with no fiber present
\( H \) = proportionality constant

\[ \text{(i.e., interfacial shear stress is proportional to mismatch between stiffness of fiber and matrix)} \]

Differentiating Eq. (6.10):
\[ \frac{d^2P}{dx^2} = \beta^2 P = -He \] (6.12) where \( \beta^2 = \frac{H}{A_fE_f} \)

Solution \( P = P_p + P_h \) (6.13)
Where \( P_p \) = particular solution = \( A_fE_fe \)
\( P_h \) = homogeneous solution
\[ = R \sinh \beta x + S \cosh \beta x \]

Resulting fiber stress is
\[ \sigma_f = \frac{P}{A_f} = E_f e \left[ \frac{\cosh \left( \frac{L}{2} - x \right)}{\cosh \left( \frac{\beta L}{2} \right)} \right] \] (6.14)

Longitudinal Modulus Vs. Fiber Length
\[ E_{f1}v_f + E_mv_m \] (3.22)
\[ E_{c1} = E_f \left[ 1 - \frac{\tanh(\beta L/2)}{\beta L/2} \right] v_f + E_m v_m \] (6.17)

as \( L \to \infty \), \( E_{c1} \to E_{f1}v_f + E_mv_m \)
as \( L \to 0 \), \( E_{c1} \to E_m v_m \)

Average fiber stress is
\[ \overline{\sigma_f} = \frac{1}{L} \int_0^L \frac{\sigma_f}{x} \, dx = \frac{E_f e}{2} \left[ 1 - \frac{\tan \left( \frac{\beta L}{2} \right) }{\beta L/2} \right] \] (6.15)

Recall “rule of mixtures” for stress under longitudinal loading,
\[ \overline{\sigma_{cl}} = \overline{\sigma_{f1}}v_f + \overline{\sigma_{m}}v_m \] (6.16)
Substituting Eq. (6.15) in (6.16) and assuming
\( \varepsilon_f = \varepsilon_m = \varepsilon_c = e \)

Longitudinal modulus is then
\[ E_{c1} = E_f \left[ 1 - \frac{\tanh(\beta L/2)}{\beta L/2} \right] v_f + E_m v_m \] (6.17)

Variation of modulus ratio \( E_{c1}/E_m \) with fiber aspect ratio, \( L/d \), for several composites.
From Gibson, et al [6.9].
Aligned Discontinuous Fiber Composite Test Specimens

Comparison of measured and predicted (Cox model) longitudinal moduli of aligned discontinuous fiber graphite/epoxy for various fiber aspect ratios. \( \frac{L}{d} \text{eff} = \frac{L}{d} \). From Suarez, et al [6.14].

Correction of Cox model to obtain better agreement with experimental data

- "Effective fiber aspect ratio"
  \[
  \left( \frac{L}{d} \right)_{\text{eff}} = Z \left( \frac{L}{d} \right)
  \]  \hspace{0.5cm} (6.20)

- Adding matrix material at ends of fiber (Fig. 6.13)
  "Modified Cox Model"
  \[
  \frac{1}{E'_{MC1}} = \frac{V_{C1}}{E_{C1}} + \frac{V_{m}}{E_{m}} = \frac{L(L+e)}{E_{C1}} + e/(L+e)
  \]  \hspace{0.5cm} (6.21)

\[E'/E'_{m}\] Vs. fiber aspect ratio for boron/epoxy without curve fitting \([E'] = 58 \times 10^6 \text{ psi} (399.62 \text{ GPa}), Z = 1, f = 51.62 \text{ Hz}\).
Finite Element Analysis

RVE:

Quarter domain: FEM mesh

Effective Modulus:

\[ E_c = \frac{\overline{\sigma}}{\overline{\varepsilon}} \]

where

\[ \overline{\sigma} = \text{average applied stress} \]
\[ \overline{\varepsilon} = \text{average strain} = \frac{\Delta}{L} \]
\[ \Delta = \text{average displacement} \]

Modified Cox model which includes matrix material at ends of fiber. From Hwang and Gibson [6.15].

Comparison of predictions from modified Cox model and finite element analysis with experimental data for boron/epoxy aligned discontinuous fiber composite at different fiber aspect ratios. From Hwang and Gibson [6.15].

Halpin – Tsai Equations

\[ \frac{E_i}{E_m} = \frac{1 + \zeta \nu_f}{1 - \nu_f} \]

(6.22)

where

\[ \eta = \left( \frac{E_{11}}{E_m} \right) - 1 \]
\[ \left( \frac{E_{11}}{E_m} \right) + \zeta \]

(6.23)

Curve – fitting parameter

\[ \zeta = 2 \frac{L}{d} \]

Assume \( E_2, G_{12}, \nu_{12} \) not significantly affected by fiber length. So, use continuous fiber models for \( E_2, G_{12}, \nu_{12} \).

Comparison of predictions from modified Cox model and finite element analysis with experimental data for boron/epoxy aligned discontinuous fiber composite at different fiber aspect ratios. From Hwang and Gibson [6.15].

Dependence of longitudinal modulus on fiber aspect ratio for aligned discontinuous fiber nylon/rubber composite. Predictions from Halpin – Tsai equations are compared with experimental results. From Halpin [6.16].
Off-axis Aligned Discontinuous Fibers

\[ E_x = \frac{1}{E_1 c'^4 + \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} c'^2 s'^2 + \frac{1}{E_2} s'^4} \]  

(2.39)

And similar equations for \( G_{xy}, \nu_{xy}, E_y \),

where \( E_1 = E_{11} = \) longitudinal modulus corrected for fiber length

But \( E_2, G_{12}, \nu_{12} \) assumed to be independent of fiber length

\[ \therefore E_x = f_1(E_{11}, E_2, G_{12}, \nu_{12}, \theta) \]

\[ E_y = f_2(E_{11}, E_2, G_{12}, \nu_{12}, \theta) \]

\[ G_{xy} = f_3(E_{11}, E_2, G_{12}, \nu_{12}, \theta) \]

\[ \nu_{xy} = f_4(E_{11}, E_2, G_{12}, \nu_{12}, \theta) \]

(6.24)

Comparison of predicted and measured off-axis modulus ratio, \( E_x/E_m \) for graphite/epoxy. From Suarez, et al [6.14].

Tridimensional plot of \( E_x/E_m \) as a function of fiber aspect ratio and fiber orientation for graphite/epoxy. From Suarez, et al [6.14].

Elastic Properties of Randomly Oriented Discontinuous Fiber Composites

- Analytical approach:
  Averaging elastic constants over all possible orientations by integration.

  Cox (1952) – analyzed a planar mat of randomly oriented continuous fibers without matrix material. Equations (6.25), (6.26) found, but not acceptable for design use.

  Nielsen and Chen (1968) – use same averaging concept, along with micromechanical equations for \( E_1, E_2, G_{12}, \nu_{12} \) and transformation equations for planar isotropic system.

Nielsen and Chen Analysis of Isotropic Young’s Modulus

\[ \bar{E} = \frac{\int E \, d\theta}{\int d\theta} \]  

(6.27)

Where \( E_x \) is given by Eq. (2.39)

\[ E_x = \frac{1}{E_1 c'^4 + \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} c'^2 s'^2 + \frac{1}{E_2} s'^4} \]

And \( E_y, E_2, G_{12}, \nu_{12} \) are found from micromechanics equations. See results in fig. 6.22

Note: fibers assumed to be continuous

Dependence of modulus ratio \( E_x/E_m \) on fiber volume fraction for several values of \( E_x/E_m \) from Nielsen – Chen model. From Nielsen and Chen [6.19].
Use of Invariants
Integration of $E_1(\theta)$ in Eq. (6.27) is cumbersome, and the use of invariants simplifies the integration process.

Example: averaged value

$$\bar{Q}_{11} = \frac{\int_{\theta} \bar{Q}_{11} d\theta}{\int_{\theta}} = \frac{\int_{\theta} [U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta] d\theta}{\pi} = U_1$$  \hspace{1cm} (6.28)

The stress–strain relationships for the planar isotropic lamina are then,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & (U_1 - U_4)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

or

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{E}/(1-\nu^2) & \nu\bar{E}/(1-\nu^2) & 0 \\ \nu\bar{E}/(1-\nu^2) & \bar{E}/(1-\nu^2) & 0 \\ 0 & 0 & \bar{E}/2(1+\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Where $\bar{E}$, $\nu$ and $\bar{G} = \frac{\bar{E}}{2(1+\nu)}$ are the engineering constants for the planar isotropic lamina.

Similarly,

$$\bar{Q}_{22} = \frac{\int_{\theta} [U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta] d\theta}{\pi} = U_1 = \bar{Q}_{11}$$

$$\bar{Q}_{12} = \bar{Q}_{21} = U_4$$

$$\bar{Q}_{16} = \frac{U_1 - U_4}{2}$$

$$\bar{Q}_{10} = \bar{Q}_{26} = 0$$

Tsai–Pagano approximations

$$\bar{E} = \frac{3}{8} E_1 + \frac{5}{8} E_2$$

$$\bar{G} = \frac{1}{8} E_1 + \frac{1}{4} E_2$$

(6.32)

Where $E_1$ and $E_2$ are estimated from micromechanics.

One approach; use Halpin–Tsai equations for aligned discontinuous Eqs. (6.22, 6.23), then use these values of $E_1$ and $E_2$ in Tsai-Pagano equations.

Comparison with exp. data for boron/epoxy in Fig. (6.23)

Manera, 1977, also showed good agreement with exp. data using Eq. (6.31)

Solving simultaneously for $\bar{E}$, $\nu$ and $\bar{G}$

(Tsai and Pagano, 1968)

$$\bar{E} = \frac{U_1 - U_4 (U_1 + U_4)}{U_1}$$

$$\nu = \frac{U_4}{U_1}$$

$$\bar{G} = \frac{U_1 - U_4}{2}$$

(6.31)

Using the definitions of the invariants $U_1$ and $U_4$

in terms of $E_1$, $E_2$, $G_{12}$ and $v_{12}$, Tsai and Pagano developed a set of approximate expressions:

Comparison of $\bar{E}$ with experimental data

(From Primer on Composite Materials, by J. C. Halpin, 1984)
Comparison of $\tilde{E} = \frac{U_i - U_1}{U_1}$ with experimental data for a glass/polyester chopped fiber composite. From M. Manera, J. Composite Materials, Vol. 11, April 1977, pp. 235 – 247

Strength of Randomly Oriented Discontinuous Fiber Composites

- Example: use maximum stress criterion to find off-axis strength, $\sigma_x$, then average over all angles.

$$\bar{\sigma}_x = \frac{2}{\pi} \left\{ \int_0^\frac{\theta_2}{2} \frac{S_L^{(s)}}{\cos \theta} d\theta + \int_\frac{\theta_2}{2}^\frac{\theta_1}{2} \frac{S_{LT}}{\sin \theta \cos \theta} d\theta + \int_\frac{\theta_1}{2}^\frac{\pi}{2} \frac{S_T^{(s)}}{\sin^2 \theta} d\theta \right\}$$

(6.47)

where,

- for $0 \leq \theta \leq \theta_1$, $\sigma_x = \frac{S_L^{(s)}}{\cos \theta}$
  (Longitudinal tensile failure)
- for $\theta_1 \leq \theta \leq \theta_2$, $\sigma_x = \frac{S_{LT}}{\sin \theta \cos \theta}$
  (Interfacial shear failure)
- for $\theta_2 \leq \theta \leq \frac{\pi}{2}$, $\sigma_x = \frac{S_T^{(s)}}{\sin^2 \theta}$
  (Transverse tensile failure)

where, $\cot \theta_1 = \frac{S_L^{(s)}}{S_{LT}}$, $\tan \theta_2 = \frac{S_T^{(s)}}{S_{LT}}$